Quantum systems in one-dimension & Quantum transport

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Phd_Course

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Why reduced dimensionality?

- Not easy to realize in condensed matter
- Effect of interactions at their strongest
- Novel physics

This is where the fun is!☺

Plan of the lectures

- Introduction to one dimensional systems

- Bosonization for fermions and Tomonaga-Luttinger liquid model
- Bosonization for bosons and examples (disordered system and Bose-glass)

- The spin chains and spin ladders (theory)

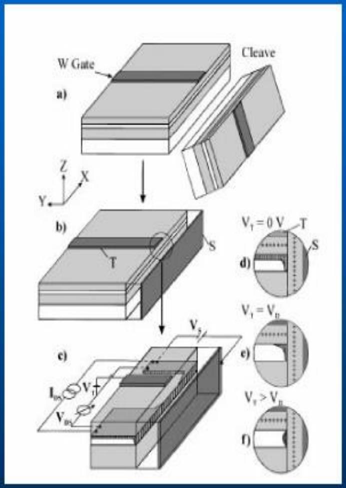
- Introduction to quantum transport
- The Landauer-Buttiker formalism for the conductance and the noise spectrum

Conclusions☺

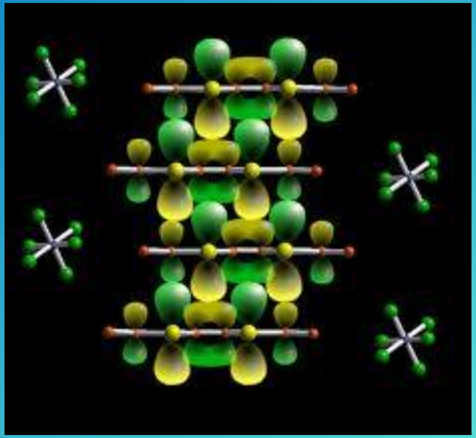
Does one dimension exist?

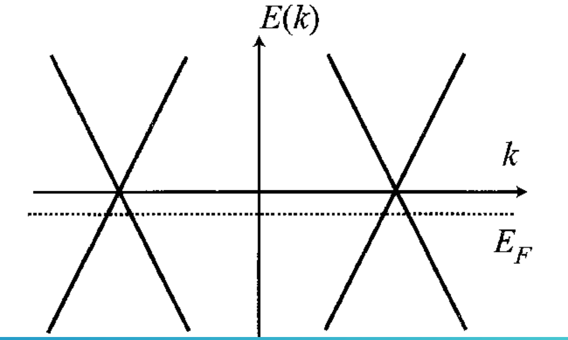
Hard to realize in condensed matter

Stacks of TM molecules



2004 First nanotube integrated circuit





Quantum wires

Organic conductors: Bechaard salt

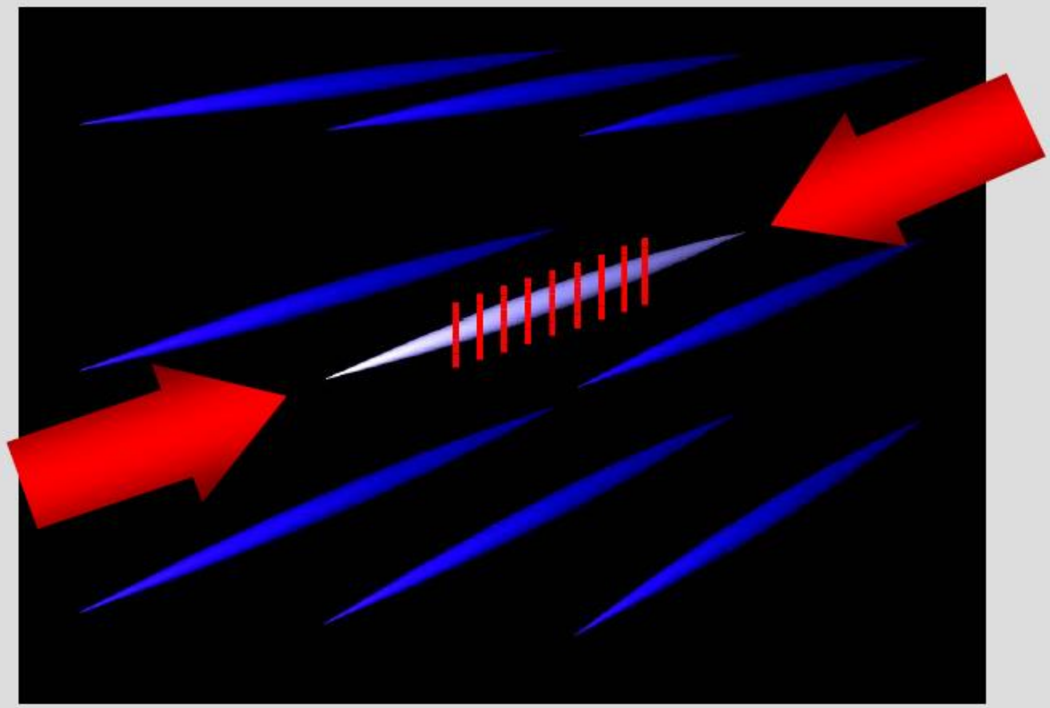
Cold atoms: an ideal system!

Optical lattices or chips

$N_0 = 10^4 - 10^5$ atoms

Creating a strongly interacting 1D Gas

^{87}Rb



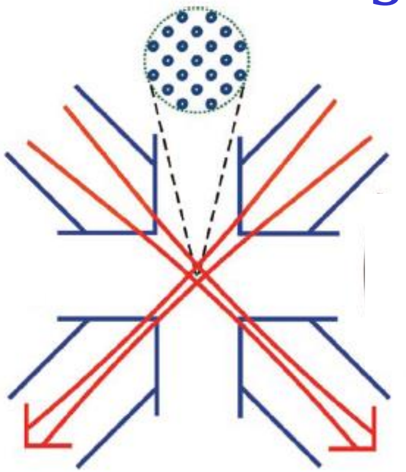
M. Greiner et al. PRL (2001)

W. Haensel et al., Nature (2002)

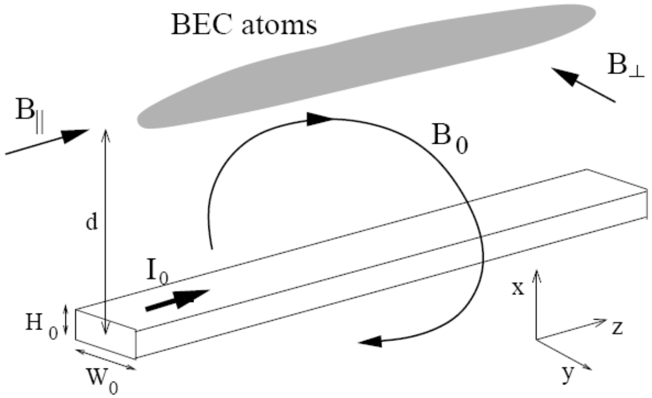
…………..

T. Stoferle et al., PRL 91 (2003), 92 (2004)

One dimensional systems: from weak to strong interactions



1D confinement in optical potential
Weiss et al., Science (05);
Bloch et al.,
Esslinger et al.,



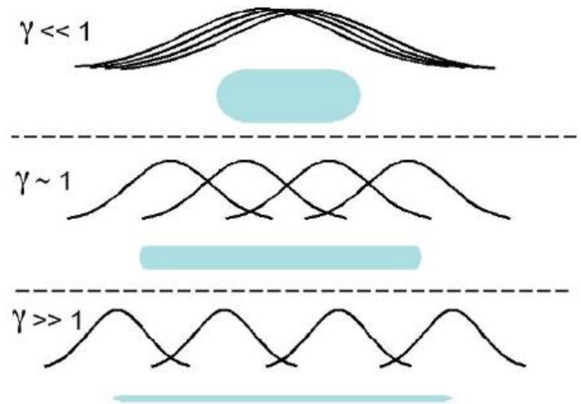
One dimensional systems in microtraps.
Thywissen et al., Eur. J. Phys. D. (99);
Hansel et al., Nature (01);
Folman et al., Adv. At. Mol. Opt. Phys. (02)
…………………………

$$\mathrm{E_{kin}} \sim \frac{\hbar^2}{m\, d^2} \sim \frac{\hbar^2\, n^2}{m}$$

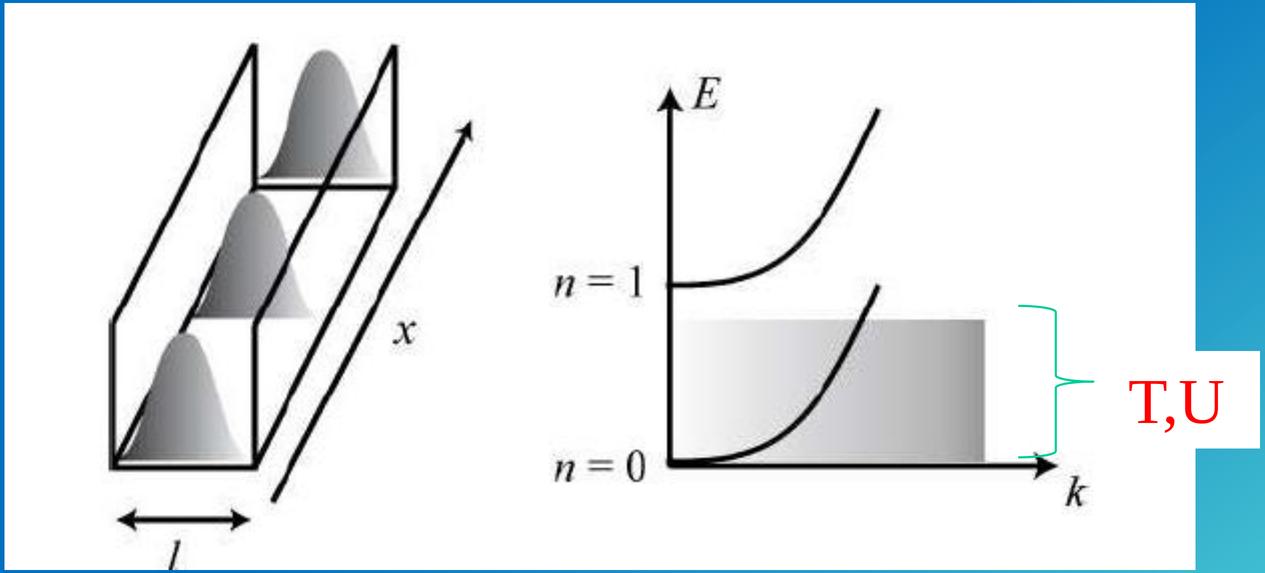
$$\mathrm{E_{int}} \sim g\, n$$

$$\gamma = \frac{\mathrm{E_{int}}}{\mathrm{E_{kin}}} \sim \frac{g\, m}{\hbar^2\, n}$$

Strongly interacting regime can be reached for low densities



What does one-dimension mean?



$$\psi(x,y) = e^{ikx}\phi(y)$$

$$\phi(y) = \sin((2n_y+1)\pi y/l)$$

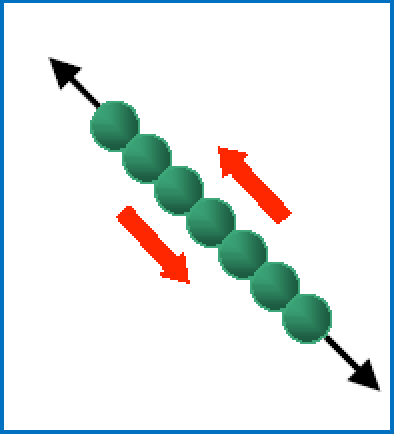
$$E = \frac{k^2}{2m} + \frac{k_y^2}{2m}$$

For hard wall confinement

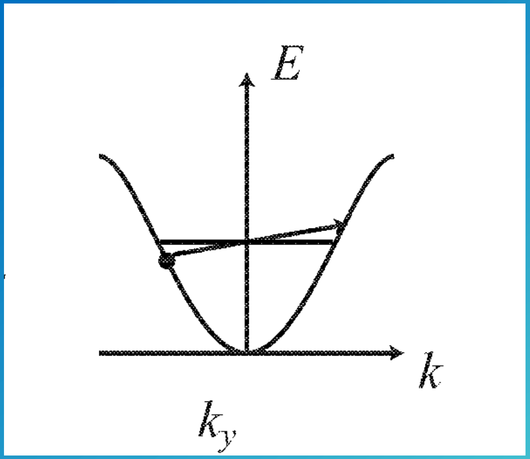
$$\Delta E = \frac{3\pi^2}{2ml^2}$$

Minimal distance between minibands

One dimension is different!



$$Q = 2k_F$$



- No individual excitations can exist only "collective" ones

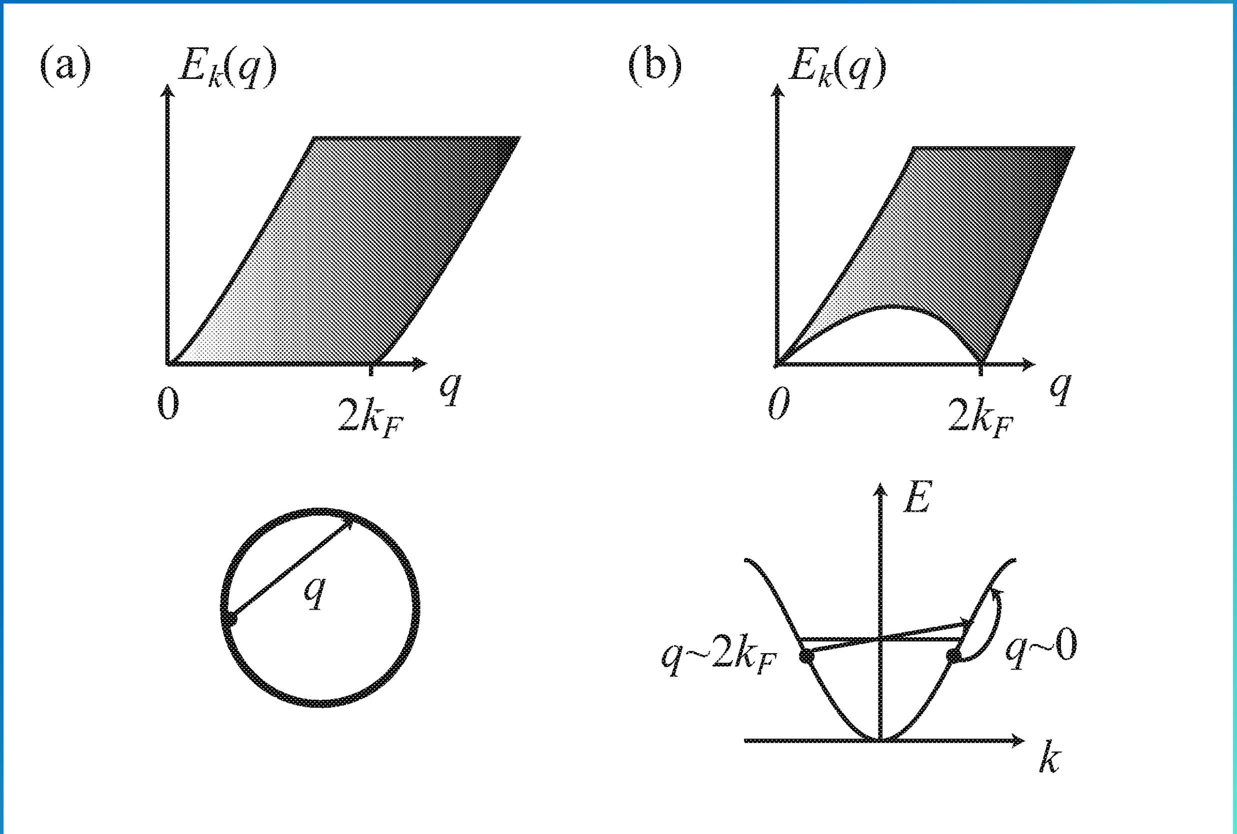
- The Fermi surface is totally nested! Strong divergences in susceptibilities

Questions:

- ❑ **How to study?**
- ✓ **Exact methods (Bethe Ansatz)** – but spectrum limited to very special models
- ✓ **Numerics—Exact**—but size limitations, quantities specific to the model
- ✓ **Low-energy methods—Bosonizatrion, smart!**

One dimension is different!

Particle-hole spectrum in 2d, 1d



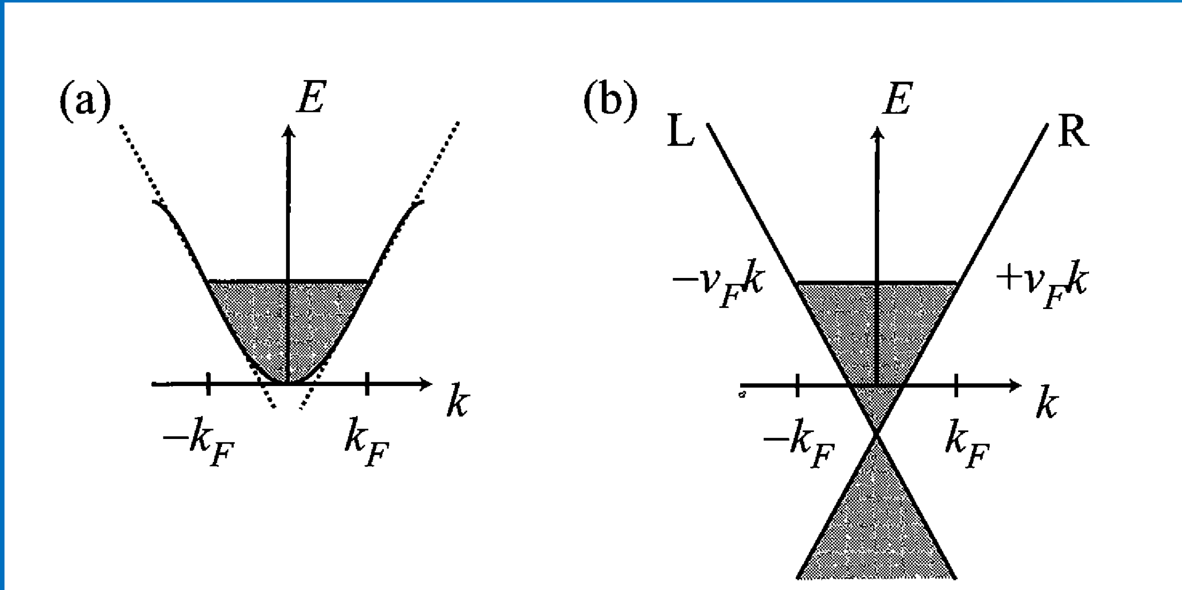
$$E_k(q) = \xi(k+q) - \xi(k)$$

$$\xi(k) = v_F(k-k_F) + \frac{\lambda}{2}(k-k_F)^2$$

$$E(q) = v_F q$$

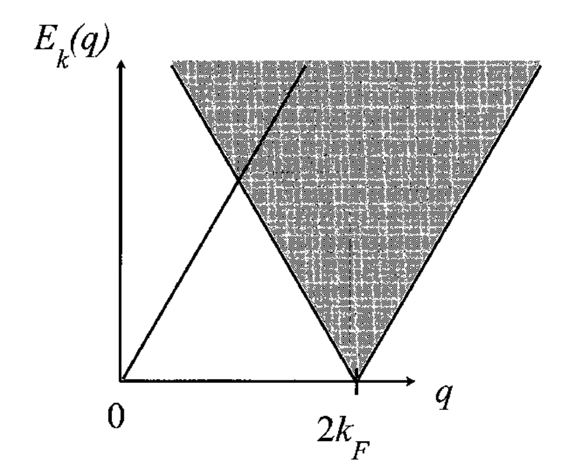
Well defined excitations

Bosonization



$$H = \sum_{k;r=R,L} v_F(\epsilon_r k - k_F) c^\dagger_{r,k} c_{r,k}$$

$$\epsilon_R = +1$$
$$\epsilon_L = -1$$



Particle-hole excitations of linearized model

$$E_{R,k}(q) = v_F(k+q) - v_F k = v_F q$$

Bosonization

Rewrite the Hamiltonian in the excitation basis: density fluctuations is a very natural basis!

$$\rho^\dagger(q) = \sum_k c^\dagger_{k+q} c_k$$

Superposition of particle-hole excitations

Quantization of density fluctuations

$$\rho(q) \sim b_q$$

$$\rho(q) \sim b^\dagger_q$$

$$\begin{aligned} H_{\text{int}} &= \frac{1}{2\Omega} \sum_q V(q) \rho(q) \rho(-q) \\ &= \frac{1}{2\Omega} \sum_q V(q) (\cdots b_q + \cdots b^\dagger_q)^2 \end{aligned}$$

Thus the interaction Hamiltonian remains quadratic in the boson basis and is simple to diagonalize! But we need to prove that excitations are boson

Only caution!

$$: AB := AB - \langle 0 | AB | 0 \rangle$$

How does it work

Different species

$$[\rho_R^\dagger(p), \rho_L^\dagger(p')] = 0$$

Equal species

$$\begin{aligned}[\rho_r^\dagger(p), \rho_r^\dagger(-p')] &= \sum_{k2}(: c^\dagger_{r,k2+p-p'} c_{r,k2} : - : c^\dagger_{r,k2-p'} c_{r,k2-p} :) \\ &\quad + \sum_{k2}(\langle 0| c^\dagger_{r,k2+p-p'} c_{r,k2}|0\rangle - \langle 0| c^\dagger_{r,k2-p'} c_{r,k2-p}|0\rangle) \\ &= \sum_{k2}(\langle 0| c^\dagger_{r,k2+p-p'} c_{r,k2}|0\rangle - \langle 0| c^\dagger_{r,k2-p'} c_{r,k2-p}|0\rangle)\end{aligned}$$

$$[\rho_r^\dagger(p), \rho_r^\dagger(-p')] = \delta_{p,p'} \sum_{k2}(\langle 0| c^\dagger_{r,k2} c_{r,k2}|0\rangle - \langle 0| c^\dagger_{r,k2-p} c_{r,k2-p}|0\rangle)$$

$$[\rho_r^\dagger(p), \rho_{r'}^\dagger(-p')] = -\delta_{r,r'}\delta_{p,p'}\frac{rpL}{2\pi}$$

$$\boxed{\begin{aligned} b_p^\dagger &= \left(\frac{2\pi}{L|p|}\right)^{1/2} \sum_r Y(rp)\rho_r^\dagger(p) \\ b_p &= \left(\frac{2\pi}{L|p|}\right)^{1/2} \sum_r Y(rp)\rho_r^\dagger(-p)\end{aligned}}$$

Bosonization dictionary

We can express the fermion operators in terms of the boson density operators

$$\psi(x) = \frac{1}{\sqrt{\Omega}} \sum_k e^{ikx} c_k \simeq \frac{1}{\sqrt{\Omega}} \sum_{k\sim -k_F} e^{ikx} c_k + \frac{1}{\sqrt{\Omega}} \sum_{k\sim k_F} e^{ikx} c_k$$
$$= \psi_L(x) + \psi_R(x)$$

$$[\rho_r^\dagger(p), \psi_r(x)] = \frac{1}{\sqrt{\Omega}} \sum_{k,k_1} e^{ik_1 x} [c^\dagger_{r,k+p} c_{r,k}, c_{r,k_1}]$$
$$= -e^{ipx} \psi_r(x)$$

$$\psi_r(x) \simeq e^{\sum_p e^{ipx} \rho_r^\dagger(-p) \left(\frac{2\pi r}{pL}\right)}$$

More convenient use the couple of operators:

$$\phi(x), \theta(x) = \mp (N_R \pm N_L) \frac{\pi x}{L} \mp \frac{i\pi}{L} \sum_{p\neq 0} \frac{1}{p} e^{-\alpha|p|/2 - ipx} (\rho_R^\dagger(p) \pm \rho_L^\dagger(p))$$

$$\boxed{\begin{aligned} \nabla\phi(x) &= -\pi[\rho_R(x) + \rho_L(x)] \\ \nabla\theta(x) &= \pi[\rho_R(x) - \rho_L(x)] \end{aligned}}$$

(for $L \to \infty$)

Bosonization dictionary

$$H = \sum_{p \neq 0} v_F |p| b_p^\dagger b_p + \frac{\pi v_F}{L} \sum_r N_r^2$$

$$\psi_r(x) = \lim_{\alpha \to 0} \frac{1}{\sqrt{2\pi\alpha}} e^{ir(k_F - \pi/L)x} e^{-i(r\phi(x) - \theta(x))}$$

$$\phi(x) = -(N_R + N_L)\frac{\pi x}{L} - \frac{i\pi}{L} \sum_p \left(\frac{L|p|}{2\pi}\right)^{1/2} \frac{1}{p} e^{-\alpha|p|/2 - ipx}(b_p^\dagger + b_{-p})$$

$$\theta(x) = (N_R - N_L)\frac{\pi x}{L} + \frac{i\pi}{L} \sum_p \left(\frac{L|p|}{2\pi}\right)^{1/2} \frac{1}{|p|} e^{-\alpha|p|/2 - ipx}(b_p^\dagger - b_{-p})$$

$$[\phi(x_1), \theta(x_2)] = i\frac{\pi}{2}\mathrm{Sign}(x_2 - x_1)$$

$$[\phi(x_1), \nabla\theta(x_2)] = i\pi\delta(x_2 - x_1)$$

$$\Pi(x) = \frac{1}{\pi}\nabla\theta(x)$$

$$\boxed{H = \frac{1}{2\pi} \int dx\, v_F [(\pi\Pi(x))^2 + (\nabla\phi(x))^2]}$$

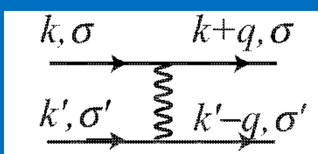
The interactions and the g-ology

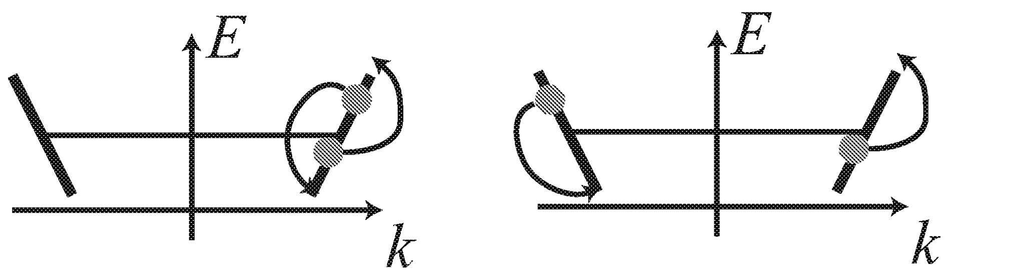
$$H_{\text{int}} = \frac{1}{2\Omega} \sum_{k,k',q} V(q) c^\dagger_{k+q} c^\dagger_{k'-q} c_{k'} c_k$$

$$\rho(x) = \frac{1}{\Omega} \sum_{k,q} e^{-iqx} c^\dagger_{k+q} c_k$$

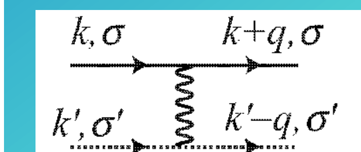
$$H = \int dx\, V(x-x')\rho(x)\rho(x')$$

$$\begin{aligned}\rho(x) &= \psi^\dagger(x)\psi(x) \\ &= \psi^\dagger_L(x)\psi_L(x) + \psi^\dagger_R(x)\psi_R(x) + \psi^\dagger_L(x)\psi_R(x) + \psi^\dagger_R(x)\psi_L(x)\end{aligned}$$

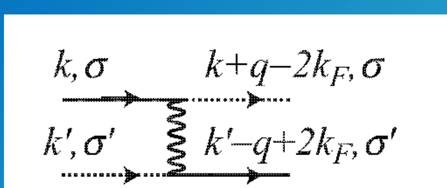


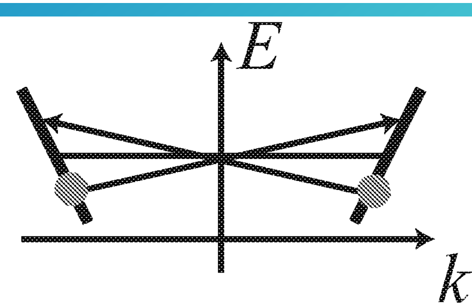


$g_4 = V(q{\sim}0)$ $g_2 = V(q{\sim}0)$



Forward scattering





$g_1 = V(q{\sim}2k_F)$

Backward scattering

For fermions with spins

$(g_\parallel, g_\perp)$

In bosonization language…

$$\frac{g_4}{2}\psi_R^\dagger(x)\psi_R(x)\psi_R^\dagger(x)\psi_R(x) = \frac{g_4}{2}\rho_R(x)\rho_R(x) = \frac{g_4}{2}\frac{1}{(2\pi)^2}(\nabla\phi - \nabla\theta)^2$$

A similar term holds for the left movers $R \to L$ and $\phi - \theta \to \phi + \theta$

$$\frac{g_4}{(2\pi)^2}\int dx[(\nabla\phi)^2 + (\nabla\theta)^2]$$

When added to the Hamiltonian this leads to a renormalization $u = v_F\left(1 + \frac{g_4}{\pi v_F}\right)$

Similarly, the interaction term g_2

$$\begin{aligned} g_2\psi_R^\dagger(x)\psi_R(x)\psi_L^\dagger(x)\psi_L(x) &= g_2\rho_R(x)\rho_L(x) \\ &= \frac{g_2}{(2\pi)^2}(\nabla\phi - \nabla\theta)(\nabla\phi + \nabla\theta) \\ &= \frac{g_2}{(2\pi)^2}[(\nabla\phi)^2 - (\nabla\theta)^2] \end{aligned}$$

The Luttinger-liquid Hamiltonian

The net effect of the interactions can be absorbed in two-parameters u,K

$$\boxed{H = \frac{1}{2\pi}\int dx[uK(\pi\Pi(x))^2 + \frac{u}{K}(\nabla\phi(x))^2]}$$

$$uK = v_F\left(1 + \frac{g_4}{2\pi v_F} - \frac{g_2}{2\pi v_F}\right)$$

$$\frac{u}{K} = \left(1 + \frac{g_4}{2\pi v_F} + \frac{g_2}{2\pi v_F}\right)v_F$$

$K < 1$ For repulsive interaction $K > 1$ For attractive interaction

Physical properties

$$C_V = \beta^2 \sum_{p\neq 0} \epsilon(p)^2 \frac{e^{\beta\epsilon(p)}}{(e^{\beta\epsilon(p)} - 1)^2} = \frac{u^2}{4T^2}\sum_{p\neq 0}\frac{p^2}{\sinh^2(\beta u p/2)} = \frac{T}{u}\left(\frac{L\pi}{3}\right)$$

$\epsilon(p) = u|p|.$

$$\kappa/\kappa_0 = v_F\frac{K}{u}$$

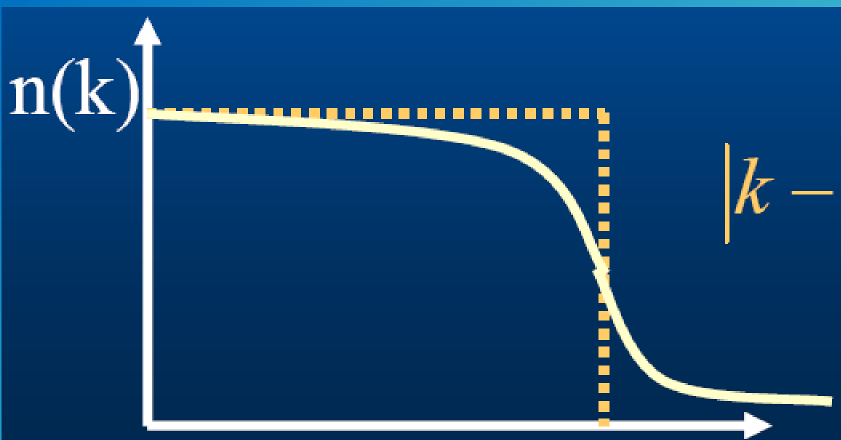
Looks like a Fermi-liquid at q≈0

Correlation functions

$$\langle \rho(x)\rho(0)\rangle = \frac{1}{x^2} + \cos(2k_F x)\left(\frac{1}{x}\right)^{2K}$$

$$\langle \psi_R(x)\psi_R^*(0)\rangle = \left(\frac{1}{x}\right)^{\frac{1}{2}[K+K^{-1}]} e^{iArg(\tau/x)}$$

$$K=1 \qquad \langle \psi_R(x)\psi_R^*(0)\rangle = \frac{1}{x - v_F\tau}$$

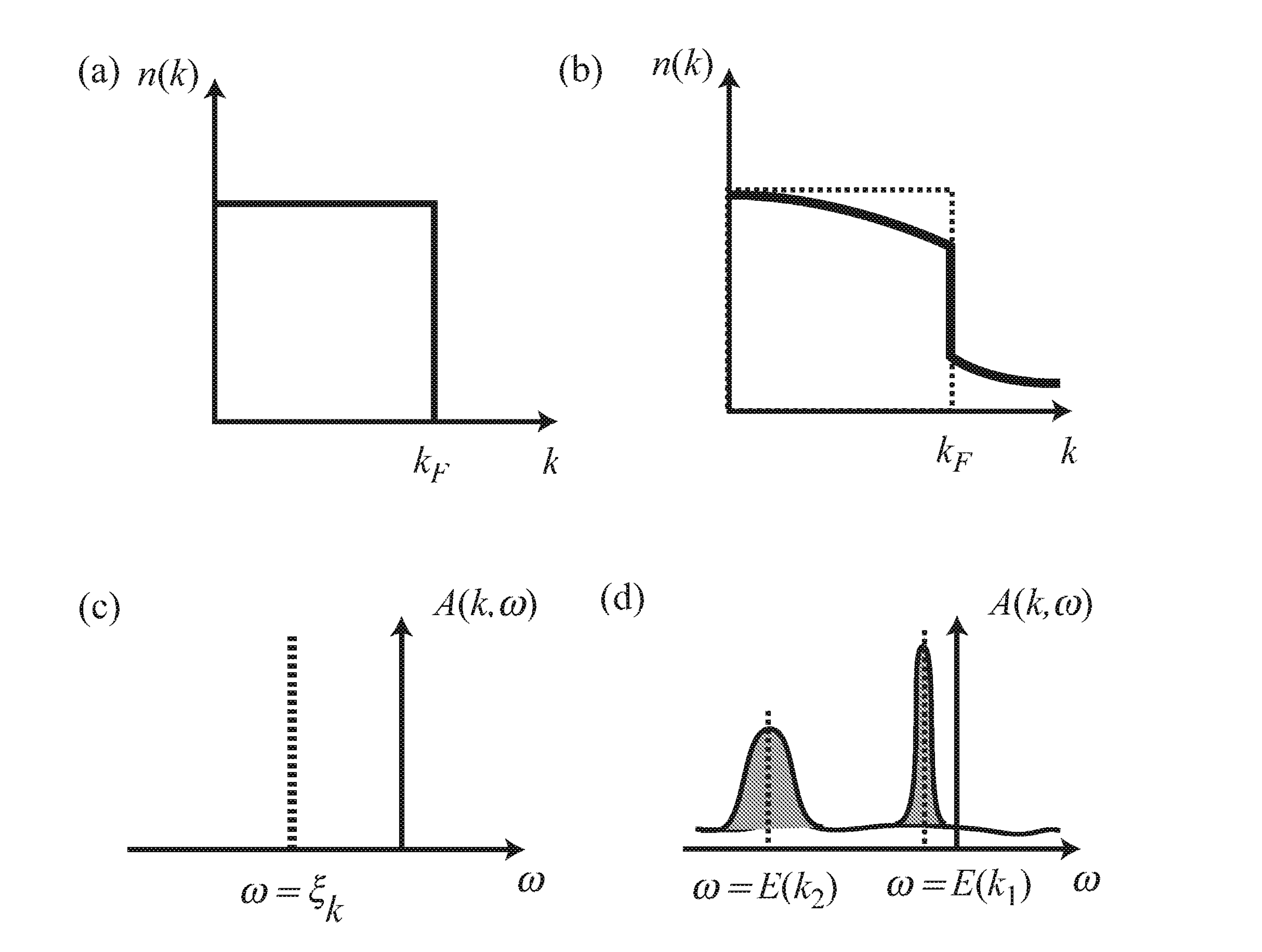


$$|k-k_F|^{\frac{1}{2}[K+K^{-1}]-1}$$

No Landau Quasiparticles !

Remind the difference: Free-fermions and Fermi-liquid

(Nozieres, 1961; Abrikosov et al, 1963; Pines and Nozieres, 1966; Mahan, 1981).



The excitations are single particles or «quasiparticles» but properties look similar!

Phase diagram in 1D: correlation functions

$$O_{\mathrm{SU}}(r) = \psi^\dagger(r)\psi^\dagger(r+a)$$

$$O_{\mathrm{SU}}(r) = \psi_R^\dagger(r)\psi_R^\dagger(r+a) + \psi_L^\dagger(r)\psi_L^\dagger(r+a) + [\psi_R^\dagger(r)\psi_L^\dagger(r+a) + \psi_L^\dagger(r)\psi_R^\dagger(r+a)]$$

The dominant contribution is coming from the last term the other are suppressed by a Pauli principle

$$O_{\mathrm{SU}}(r) \to \frac{2}{2\pi\alpha}[e^{-i2\theta(r)}\cos(2\phi(r) - 2k_F x) + e^{-i2\theta(r)}]$$

$$O_{\mathrm{SU}}(r) \simeq \frac{e^{-i2\theta(r)}}{\pi\alpha}$$

Superconducting correlations

$$R_{\mathrm{SU}}(r) = \langle O_{\mathrm{SU}}(r) O_{\mathrm{SU}}^\dagger(0)\rangle = \frac{1}{(\pi\alpha)^2}\left(\frac{\alpha}{r}\right)^{1/(2K)}$$

Superconducting correlations decay very slowly, as a power-law with non-universal exponent. The tendency of the system to have stronger SC fluctuations is when the decay is slower, i.e. for K large.

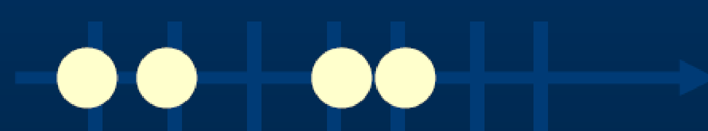
Phase diagram

$$\chi(q,\omega) = \int dx d\tau \; e^{i(qx+\omega\tau)} \chi(x,\tau) \qquad \chi \approx \omega^{\eta-2}$$

Most divergent fluctuations

K<1 K>1 -V

CDW SU



System with spin

Same treatment

$$\rho_\uparrow \rightarrow \nabla\Phi_\uparrow \qquad \rho_\downarrow \rightarrow \nabla\Phi_\downarrow$$

More convenient

$$\rho = \frac{1}{\sqrt{2}}(\rho_\uparrow + \rho_\downarrow) \qquad \sigma = \frac{1}{\sqrt{2}}(\rho_\uparrow - \rho_\downarrow)$$

$$H_{kin} = H_\uparrow + H_\downarrow = H_\rho + H_\sigma$$

$$H_{\text{int}} = U\sum_i \rho_\uparrow \rho_\downarrow = U(\rho+\sigma)(\rho-\sigma)$$

$$= U(\rho\rho - \sigma\sigma)$$

$$H = H_\rho + H_\sigma$$

(u_ρ, K_ρ) Charge excitations

(u_σ, K_σ) Spin excitations

Charge-spin separation

Total Hamiltonian: charge+spin

$$H = H_\rho + H_\sigma + \frac{2g_1}{(2\pi\alpha)^2}\int dx \cos(\sqrt{8}\phi_\sigma) \quad .$$

$$H_\nu = \int dx \left(\frac{\pi u_\nu K_\nu}{2}\Pi_\nu^2 + \frac{u_\nu}{2\pi K_\nu}(\partial_x \phi_\nu)^2 \right)$$

$\nu = \rho, \sigma$

$$u_\nu = \sqrt{\left(v_F + \frac{g_{4,\nu}}{\pi}\right)^2 - \left(\frac{g_\nu}{2\pi}\right)^2} \ , \ K_\nu = \sqrt{\frac{2\pi v_F + 2g_{4,\nu} + g_\nu}{2\pi v_F + 2g_{4,\nu} - g_\nu}} \ ,$$

Spin sector more complicated (gap)

$$H = \int \frac{dx}{2\pi}[uK(\pi\Pi(x))^2 + \frac{u}{K}(\nabla\Phi(x))^2]$$
$$+ g\int dx \cos(\sqrt{8}\Phi(x))$$

Anomalous correlation functions

$$n(k) \approx \left|k - k_F\right|^{\frac{1}{4}[K_\rho + K_\rho{}^{-1}] - \frac{1}{2}}$$ photoemission

$$\chi_{2k_F} \approx T^{K_\rho - 1}$$ NMR

Physical quantities

Compressibility

$$H = -\mu \int dx [\rho_\uparrow(x) + \rho_\downarrow(x)] = \frac{\mu\sqrt{2}}{\pi} \int dx \, \nabla\phi_\rho(x)$$

$$\kappa_\rho = \frac{2K_\rho}{\pi u_\rho}$$

Uniform magnetic susceptibility

$$H = -\mathbf{h} \int dx \frac{1}{2} [\rho_\uparrow(x) - \rho_\downarrow(x)] = \frac{\mathbf{h}}{\pi\sqrt{2}} \int dx \, \nabla\phi_\sigma(x)$$

$$\kappa_\sigma = \frac{K_\sigma}{2\pi u_\sigma}$$

Spin and density correlations

$$O_\rho(x) = \sum_{\sigma,\sigma'} \psi_\sigma^\dagger(x) \delta_{\sigma,\sigma'} \psi_{\sigma'}(x)$$

$$O_\sigma^a(x) = \sum_{\sigma,\sigma'} \psi_\sigma^\dagger(x) \sigma_{\sigma,\sigma'}^a \psi_{\sigma'}(x)$$

$$O_\rho(x) = \frac{-\sqrt{2}}{\pi} (\nabla\phi_\rho(x)) + (O_{\mathrm{CDW}}(x) + \mathrm{h.c.})$$

$$O_\sigma^z(x) = \frac{-\sqrt{2}}{\pi} (\nabla\phi_\sigma(x)) + (O_{\mathrm{SDW}}^z(x) + \mathrm{h.c.})$$

$2k_F$ components

Spin and charge DW

$$O_{\mathrm{CDW}}(x) = \psi^\dagger_{R\uparrow}\psi_{L\uparrow}(x) + \psi^\dagger_{R\downarrow}\psi_{L\downarrow}(x) = \frac{e^{-2ik_F x}}{\pi\alpha} e^{i\sqrt{2}\phi_\rho}\cos(\sqrt{2}\phi_\sigma)$$

$$O^x_{\mathrm{SDW}}(x) = \psi^\dagger_{R\uparrow}\psi_{L\downarrow}(x) + \psi^\dagger_{R\downarrow}\psi_{L\uparrow}(x) = \frac{e^{-2ik_F x}}{\pi\alpha} e^{i\sqrt{2}\phi_\rho}\cos(\sqrt{2}\theta_\sigma)$$

$$O^y_{\mathrm{SDW}}(x) = -i(\psi^\dagger_{R\uparrow}\psi_{L\downarrow}(x) - \psi^\dagger_{R\downarrow}\psi_{L\uparrow}(x)) = \frac{-e^{-2ik_F x}}{\pi\alpha} e^{i\sqrt{2}\phi_\rho}\sin(\sqrt{2}\theta_\sigma)$$

$$O^z_{\mathrm{SDW}}(x) = \psi^\dagger_{R\uparrow}\psi_{L\uparrow}(x) - \psi^\dagger_{R\downarrow}\psi_{L\downarrow}(x) = \frac{e^{-2ik_F x}}{\pi\alpha} e^{i\sqrt{2}\phi_\rho} i\sin(\sqrt{2}\phi_\sigma)$$

$$\langle O^\dagger_{\mathrm{CDW}}(r) O_{\mathrm{CDW}}(0)\rangle = \frac{e^{2ik_F x}}{2(\pi\alpha)^2}\left(\frac{\alpha}{r}\right)^{K_\rho + K_\sigma}$$

$$\langle O^{x\dagger}_{\mathrm{SDW}}(r) O^x{}_{\mathrm{SDW}}(0)\rangle = \frac{e^{2ik_F x}}{2(\pi\alpha)^2}\left(\frac{\alpha}{r}\right)^{K_\rho + 1/K_\sigma}$$

$$\langle O^{y\dagger}_{\mathrm{SDW}}(r) O^y{}_{\mathrm{SDW}}(0)\rangle = \frac{e^{2ik_F x}}{2(\pi\alpha)^2}\left(\frac{\alpha}{r}\right)^{K_\rho + 1/K_\sigma}$$

$$\langle O^{z\dagger}_{\mathrm{SDW}}(r) O^z{}_{\mathrm{SDW}}(0)\rangle = \frac{e^{2ik_F x}}{2(\pi\alpha)^2}\left(\frac{\alpha}{r}\right)^{K_\rho + K_\sigma}$$

$K_\sigma \neq 1$ Breaks rotation symmetry

Pairing operators

$$O_{\mathrm{SS}}(x) = \sum_{\sigma,\sigma'} \sigma \psi^\dagger_{R,\sigma}(x) \delta_{\sigma,\sigma'} \psi^\dagger_{L,-\sigma'}(x)$$

$$O^a_{\mathrm{TS}}(x) = \sum_{\sigma,\sigma'} \sigma \psi^\dagger_{R,\sigma}(x) \sigma^a_{\sigma,\sigma'} \psi^\dagger_{L,-\sigma'}(x)$$

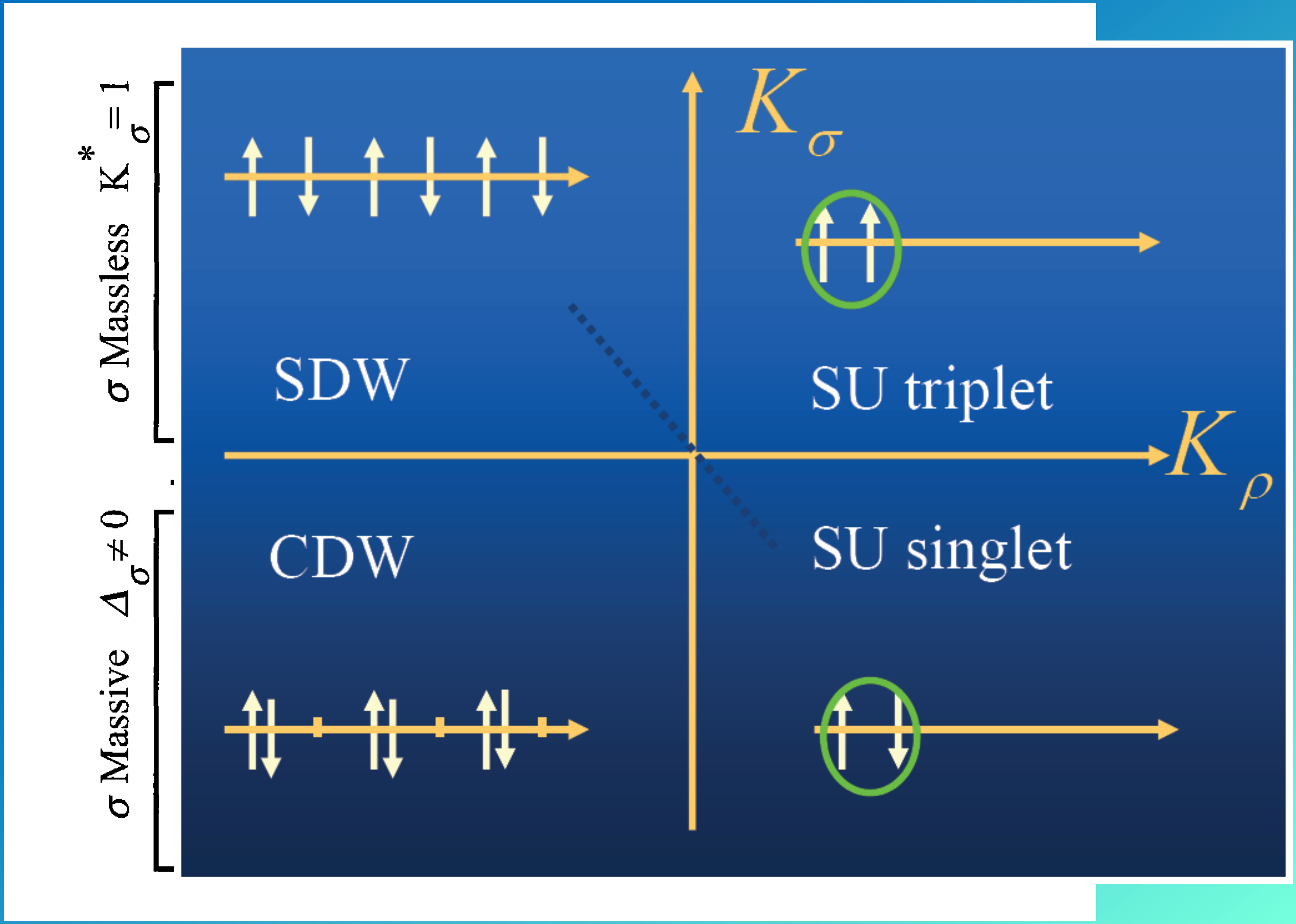
$$O_{\mathrm{SS}}(x) = \psi^\dagger_{R\uparrow}\psi^\dagger_{L\downarrow}(x) + \psi^\dagger_{L\uparrow}\psi^\dagger_{R\downarrow}(x) = \frac{1}{\pi\alpha} e^{-i\sqrt{2}\theta_\rho} \cos(\sqrt{2}\phi_\sigma)$$

$$O^x_{\mathrm{TS}}(x) = \psi^\dagger_{R\uparrow}\psi^\dagger_{L\uparrow}(x) + \psi^\dagger_{L\downarrow}\psi^\dagger_{R\downarrow}(x) = \frac{1}{\pi\alpha} e^{-i\sqrt{2}\theta_\rho} \cos(\sqrt{2}\theta_\sigma)$$

$$O^y_{\mathrm{TS}}(x) = -i(\psi^\dagger_{R\uparrow}\psi^\dagger_{L\uparrow}(x) - \psi^\dagger_{L\downarrow}\psi^\dagger_{R\downarrow}(x)) = \frac{-1}{\pi\alpha} e^{-i\sqrt{2}\theta_\rho} \sin(\sqrt{2}\theta_\sigma)$$

$$O_{\mathrm{TS}}(x) = \psi^\dagger_{R\uparrow}\psi^\dagger_{L\downarrow}(x) - \psi^\dagger_{L\uparrow}\psi^\dagger_{R\downarrow}(x) = \frac{e^{2ik_F x}}{\pi\alpha} e^{-i\sqrt{2}\theta_\rho} \sin(\sqrt{2}\phi_\sigma)$$

The phase diagram



The lattice effect

Existence of a lattice means that a wv is defined modulo a wv of the reciprocal lattice

$2\pi/a$

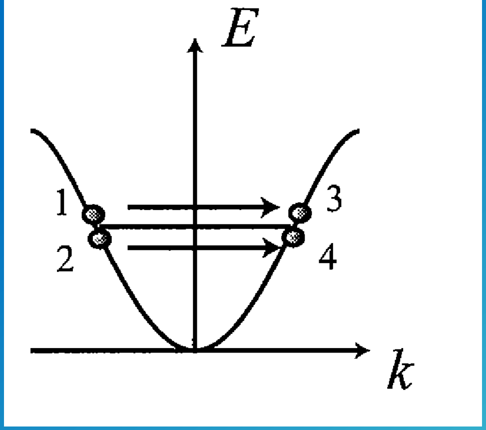
$k_1 + k_2 - k_3 - k_4 = Q$

These processes are possible

$k_1 + k_2 - k_3 - k_4 = 2\pi$

$4k_F = 2\pi$

i.e. half-filling



Umklapp processes

$$H = \int dx\, U \rho_\uparrow(x)\rho_\downarrow(x)$$

$$H = \int dx\, U \frac{1}{(2\pi\alpha)^2}[e^{-i4k_F x} e^{i2(\phi_\uparrow(x)+\phi_\downarrow(x))} + \text{h.c.}]$$

The 4kF term is non oscillating!

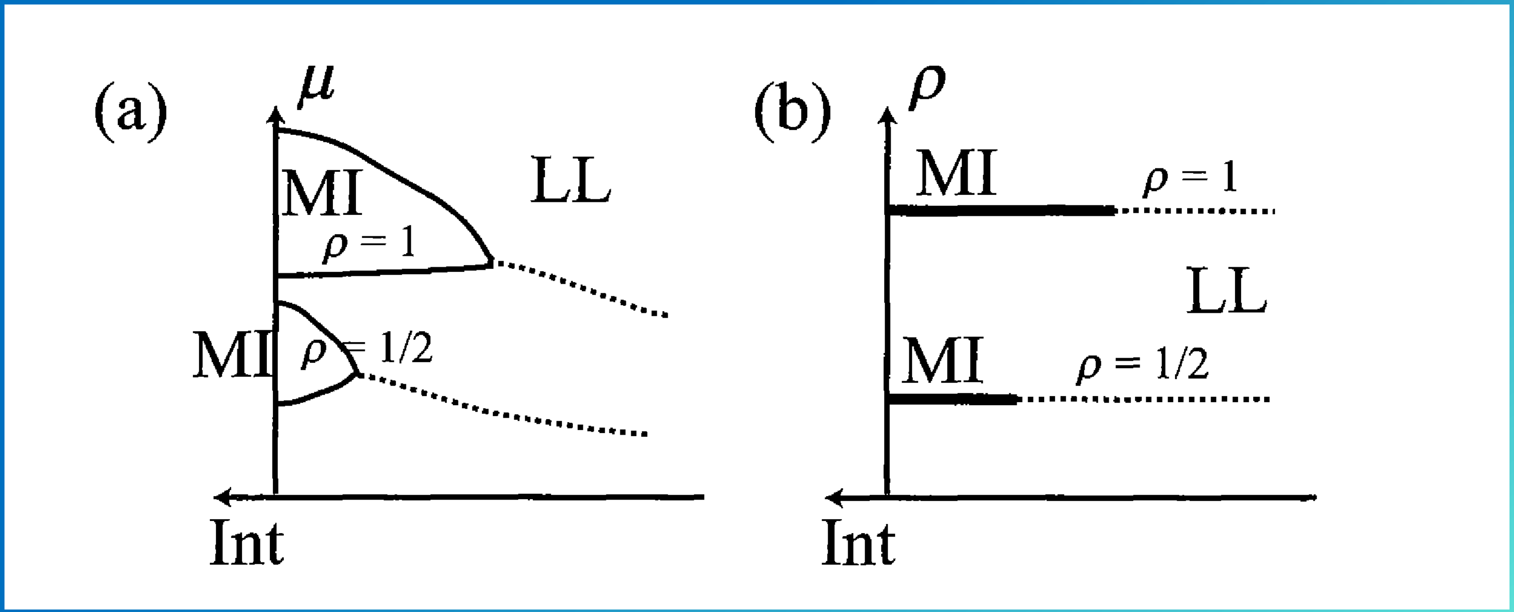
$$H = H_0 + \frac{2g_3}{(2\pi\alpha)^2} \int dx\ \cos(\sqrt{8}\phi_\rho(x))$$

So the charge part is sine-Gordon type

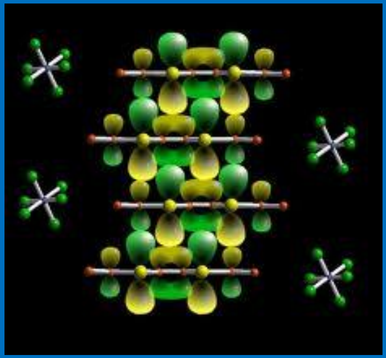
The term survive for other commensurability too

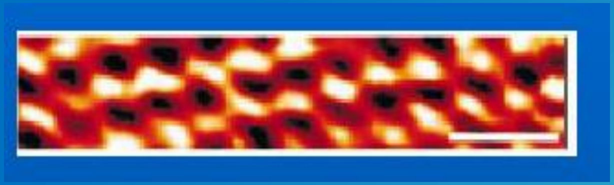
$2pk_F = 2\pi q/a$

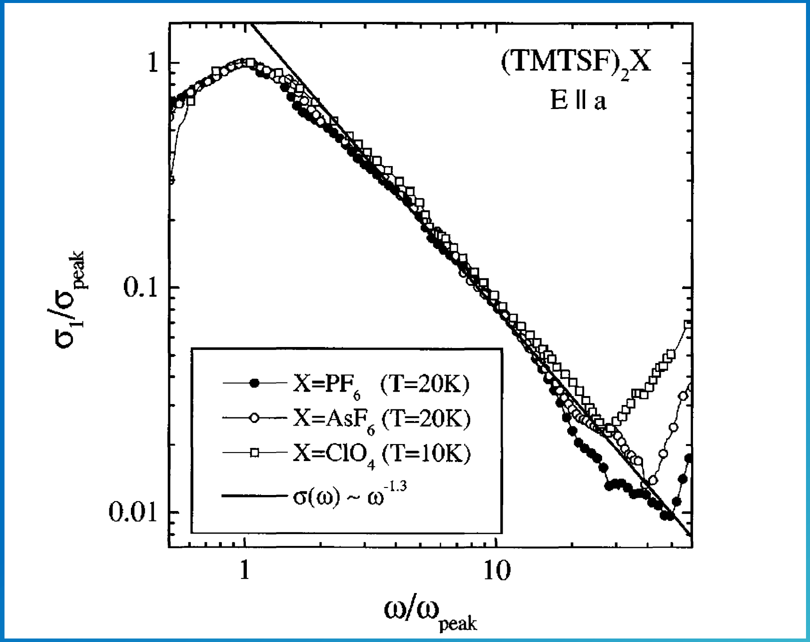
Phase diagram: The Mott-insulator

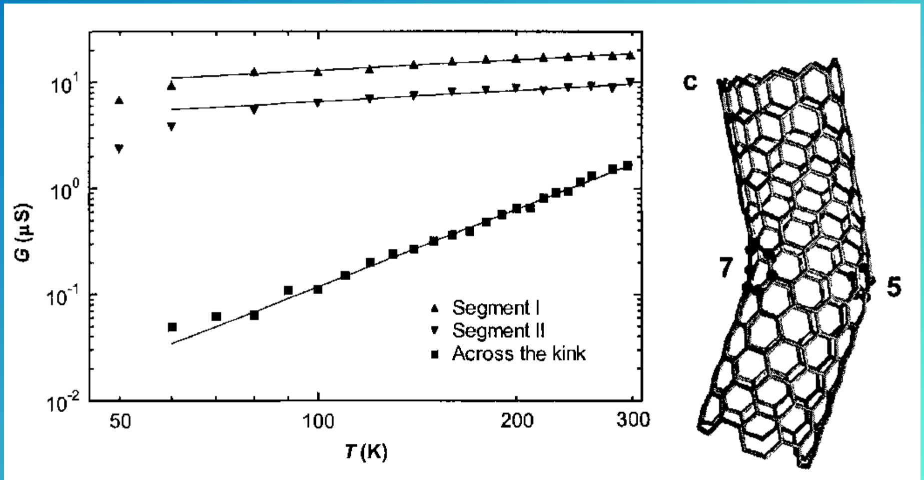


Check for the power laws:









Yao et. Al. Nature 402, 1999

A. Schwartz, PRB 58, 1998

References

✓**T. Giamarchi**, Quantum Physics in one Dimension. vol. 121, (Oxford University Press, Oxford, UK, 2004).

✓ **F. D. M. Haldane**, Effective harmonic-fluid approach to low-energy properties of one dimensional quantum fluids, Physical Review Letters. 47, 1840, (1981).

✓ **M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol**, One dimensional bosons: From condensed matter systems to ultracold gases, Reviews of Modern Physics. 83, 1405, (2011).
doi: 10.1103/RevModPhys.83.1405

Quantum systems in one-dimension & Quantum transport

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_Phd_Course_

IPCMS, Srasbourg

_08-11-2016_

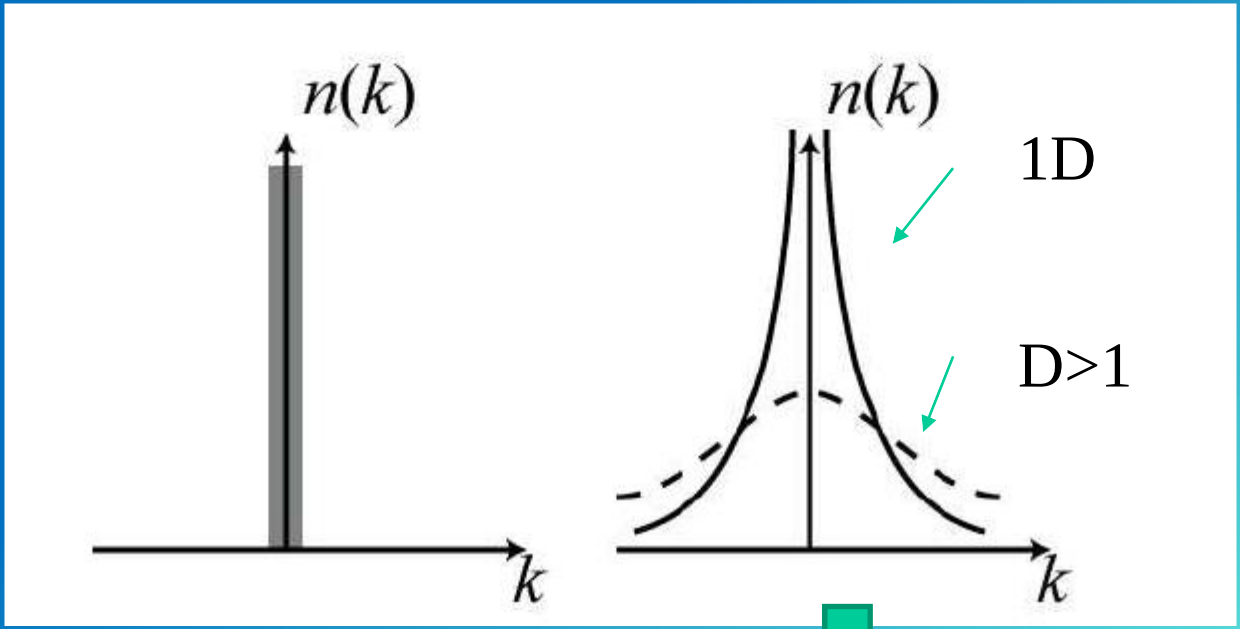
Plan of the lectures

- Introduction to one dimensional systems
- Bosonization for fermions and Luttinger liquid physics
- Bosonization for bosons and examples
- The spin chains and spin ladders (theory)
- Introduction to quantum transport
- The Landauer-Buttiker formalism for the conductance and the noise spectrum

Conclusions☺

Free bosons: crash course

For free particles at d>1: condensation in the k-space at k=0



$$H_0 = \sum_k \varepsilon_k b_k^+ b_k$$

$$n_k = \langle b_k^+ b_k \rangle$$

Effect of interactions

Models

Continuum:

$$H = \int dx \frac{\hbar^2 (\nabla\psi)^\dagger (\nabla\psi)}{2m} + \frac{1}{2}\int dx\, dx'\, V(x-x')\rho(x)\rho(x') - \mu_0 \int dx\, \rho(x)$$

$$V(x) = V_0 \delta(x)$$

Lieb, E. H., and Liniger,W., *Phys. Rev.*, ***130*, *1605 (1963)*.**

Lattice:

$$H = -t\sum_i (b^\dagger_{i+1} b_i + \mathrm{h.c.}) + U\sum_i n_i(n_i-1) - \sum_i \mu_i n_i$$

Within a tight-binding approximation

$$t = \langle \psi_0(x+a)|H_{\mathrm{kin}}|\psi_0(x)\rangle$$

$$U = \int dxdydz |\psi(x,y,z)|^4$$

$$\psi_0(x) = \left(\frac{m\omega_0}{\hbar\pi}\right)^{1/4} e^{-\frac{m\omega_0}{2\hbar}x^2}$$

$$\omega_0^2 = 4V_L k^2$$

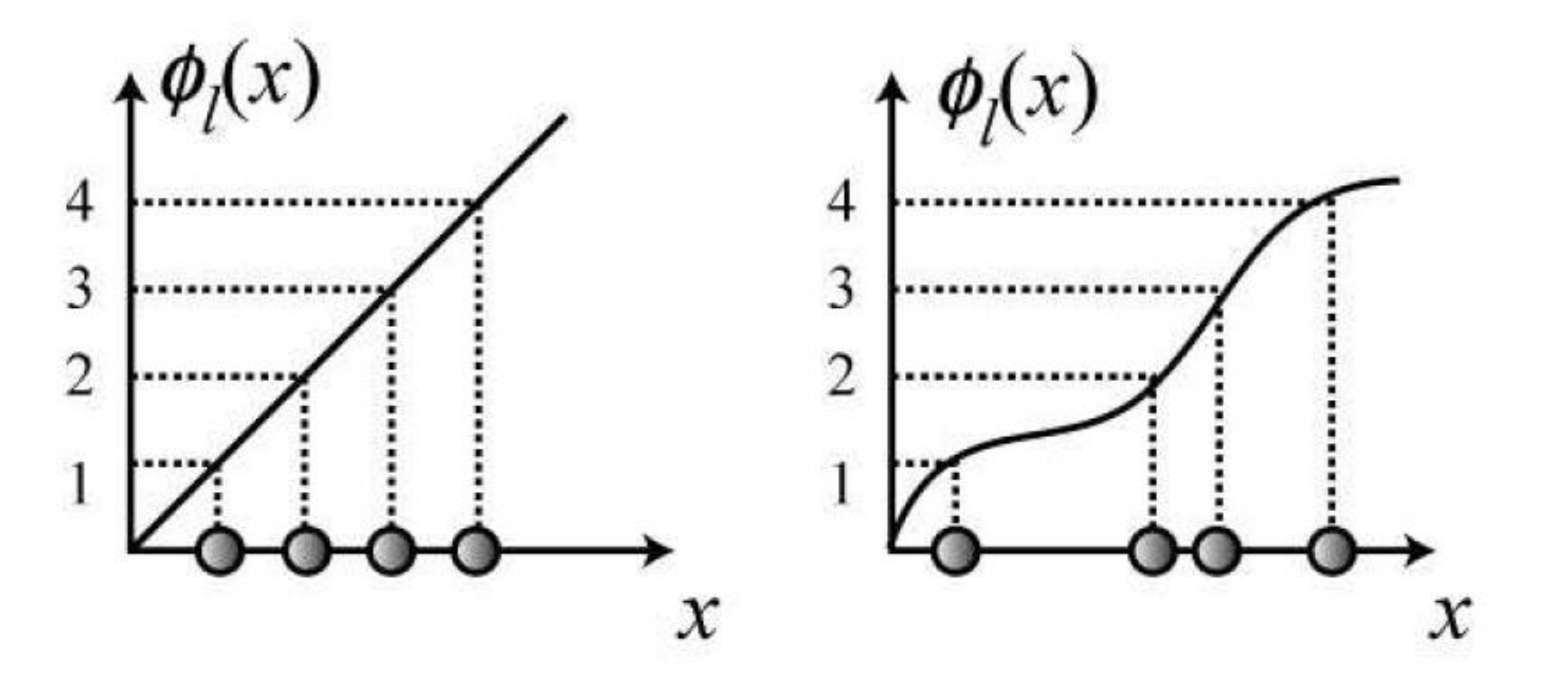
$$\psi(x,y,z) = \psi_0(x)\psi_\perp(y)\psi_\perp(z)$$

Labelling the particles: bosonization

The idea behind the bosonization technique is to reexpress the excitations of the system in a basis of collective excitations

$$\begin{aligned}\rho(x) &= \sum_i \delta(x - x_i) \\ &= \sum_n |\nabla\phi_l(x)|\,\delta(\phi_l(x) - 2\pi n)\end{aligned}$$

$$\rho(x) = \frac{\nabla\phi_l(x)}{2\pi} \sum_p e^{ip\phi_l(x)}$$



1D unique way of labelling!

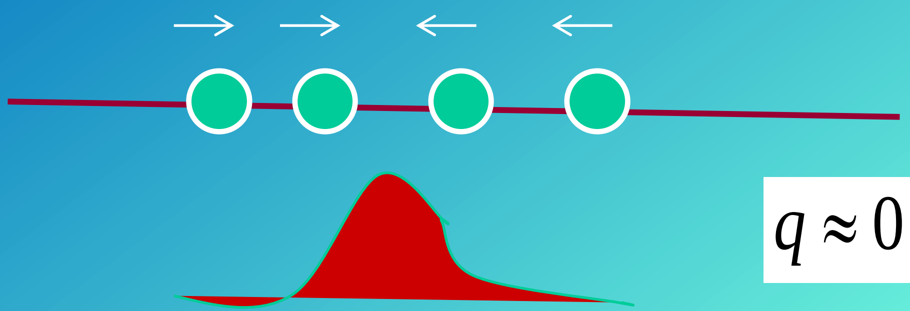
With respect to crystalline situation

$$\phi_l(x) = 2\pi\rho_0 x - 2\phi(x)$$

$$\rho(x) = \left[\rho_0 - \frac{1}{\pi}\nabla\phi(x)\right]\sum_p e^{i2p(\pi\rho_0 x - \phi(x))}$$

$\phi(x)$ varies slowly thus

$$\boxed{\rho_{q\sim 0}(x) \simeq \rho_0 - \frac{1}{\pi}\nabla\phi(x)}$$



$q \approx 0$

Bosonization:

$$\psi^{\dagger}(x) = [\rho(x)]^{1/2} e^{-i\theta(x)}$$

θ superfluid phase

$$\rho(x) = \left[\rho_0 - \frac{1}{\pi}\nabla\phi(x)\right] \sum_p e^{i2p(\pi\rho_0 x - \phi(x))}$$

$$[\frac{1}{\pi}\nabla\phi(x), \theta(x')] = -i\delta(x - x')$$

Quantum fluctuations

$$\psi_B^{\dagger}(x) = [\rho_0 - \frac{1}{\pi}\nabla\phi(x)]^{1/2} \sum_p e^{i2p(\pi\rho_0 x - \phi(x))} e^{-i\theta(x)}$$

The hamiltonian

$$H_K \simeq \int dx \frac{\hbar^2\rho_0}{2m}(\nabla e^{i\theta})(\nabla e^{-i\theta}) = \int dx \frac{\hbar^2\rho_0}{2m}(\nabla\theta)^2$$

$$H_{\text{int}} = \int dx V_0 \frac{1}{2\pi^2}(\nabla\phi)^2$$

The Tomonaga-Luttinger liquid:

$$H = \frac{\hbar}{2\pi} \int dx [\frac{uK}{\hbar^2} (\pi \Pi(x))^2 + \frac{u}{K} (\nabla \phi(x))^2]$$

Haldane, F. D. M., *J. Phys. C*, ***14, 2585 (1981)***

Cazalilla et al, *Rev. Mod. Phys*., **83** (***2011***)

$$S/\hbar = \frac{1}{2\pi K} \int dx\, d\tau [\frac{1}{u} (\partial_\tau \phi)^2 + u (\partial_x \phi(x))^2]$$

Standard sound wave Hamiltonian with relation dispersion:

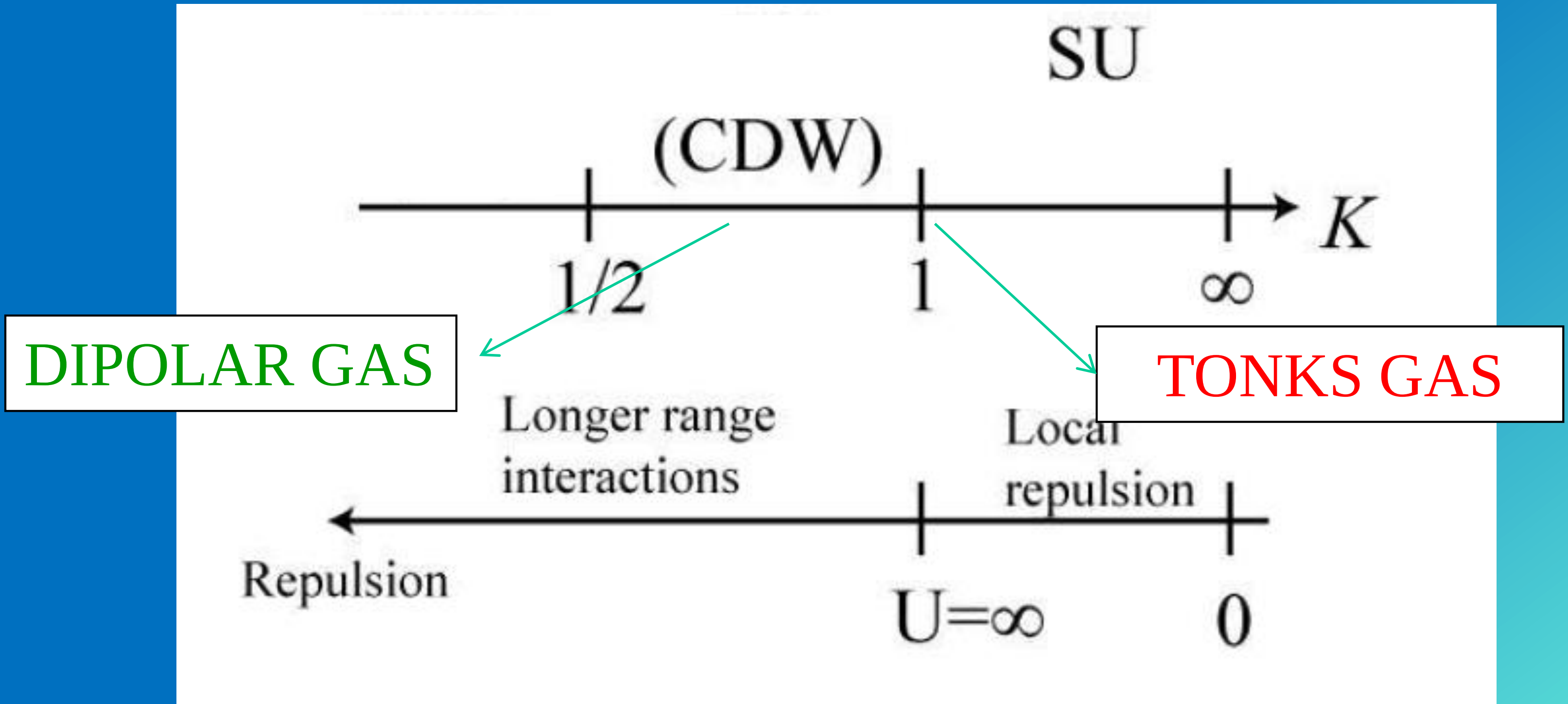
$$\omega^2 = u^2 k^2$$

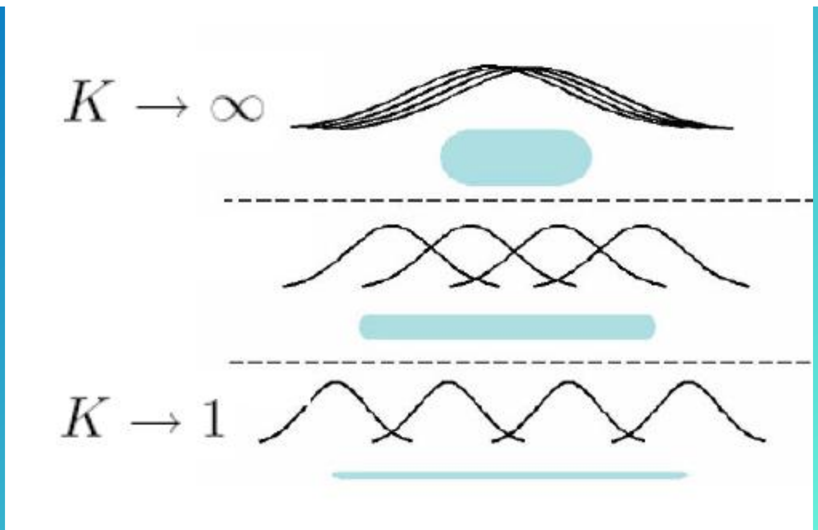
$$uK = \frac{\pi \hbar \rho_0}{m}$$

$$\frac{u}{K} = \frac{V_0}{\hbar \pi}$$

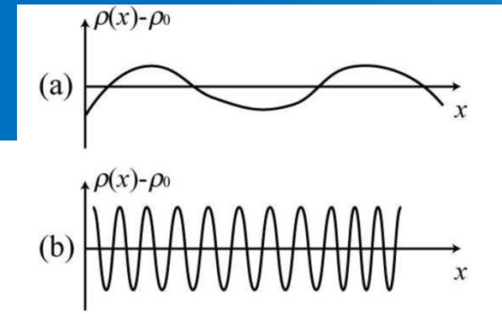
Luttinger liquid: low energy properties (fermions, bosons, spins..), fixed point of massless theory

Phase diagram as a function of K:



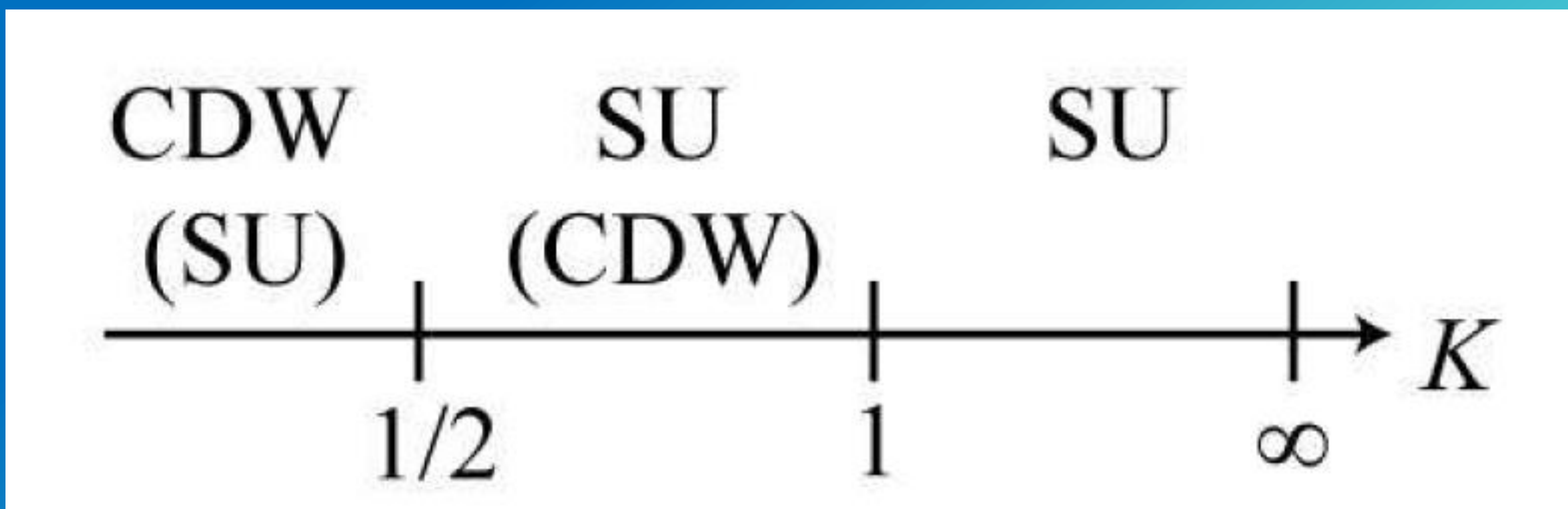


Correlations



$$\langle T_\tau \psi(r)\psi^\dagger(0)\rangle = A_1 \left(\frac{\alpha}{r}\right)^{\frac{1}{2K}} + \cdots$$

$$\langle T_\tau \rho(r)\rho(0)\rangle = \rho_0^2 + \frac{K}{2\pi^2}\frac{y_\alpha^2 - x^2}{(y_\alpha^2 + x^2)^2} + A_3 \cos(2\pi\rho_0 x)\left(\frac{1}{r}\right)^{2K} + \cdots$$

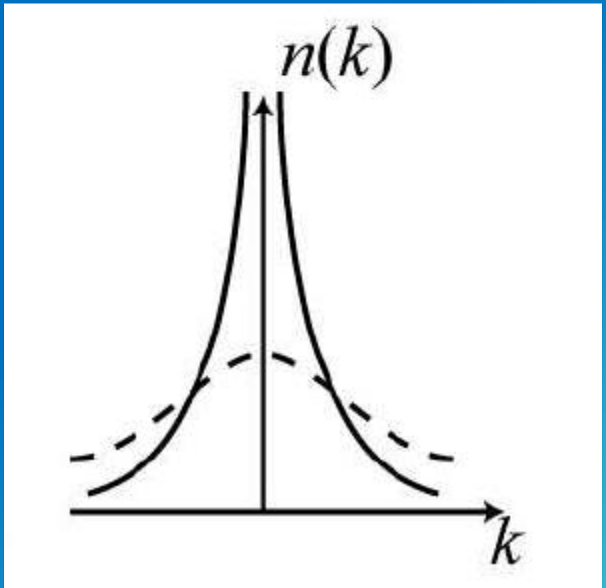


Condensate?

No true condensate!

$G(x \to \infty, \tau = 0) \to |\psi_0|^2$

Only in the presence of a true condensate



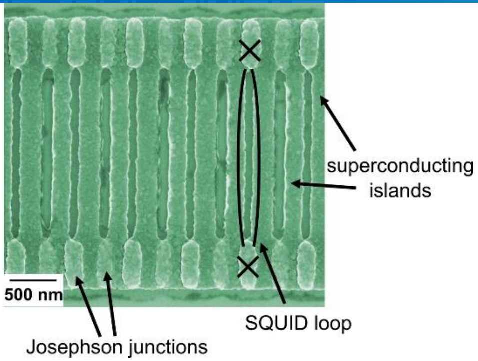
$G(x,\tau) = \langle T_\tau \psi(x,\tau)\psi^\dagger(0,0)\rangle$ $\longrightarrow$ $n(k) \propto k^{\frac{1}{2K}-1}$

as follows from the FT of the generalized susceptilility

$\chi(\omega_n, k) = \int dx d\tau \chi(x,\tau)$ $\chi(r) \sim (1/r)^\mu$

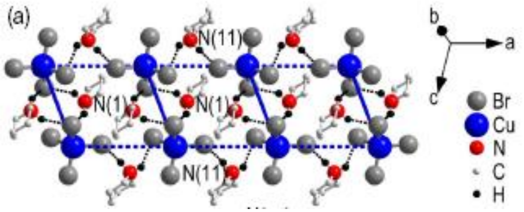
Experimental boson systems realizing Luttinger liquid physics

- ✓ Josephson junctions

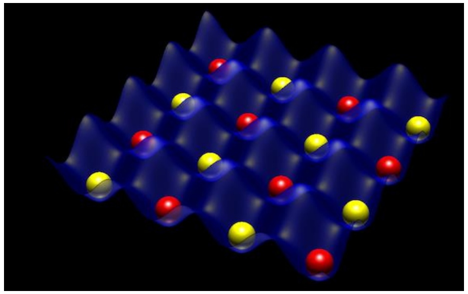


- ✓ Two-leg ladders in a magnetic field

BPCP



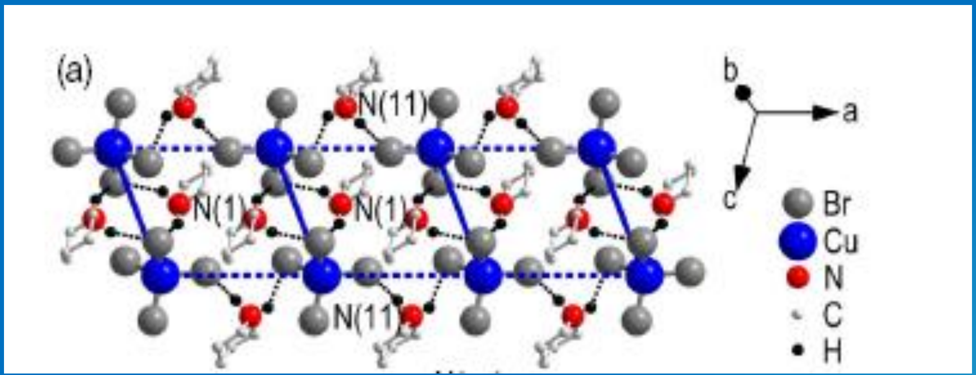
- ✓ Trapped atoms



PRL 101, 137207 (2008) PHYSICAL REVIEW LETTERS week ending 26 SEPTEMBER 2008

Controlling Luttinger Liquid Physics in Spin Ladders under a Magnetic Field

M. Klanjšek,[1] H. Mayaffre,[2] C. Berthier,[1] M. Horvatić,[1] B. Chiari,[3] O. Piovesana,[3] P. Bouillot,[4] C. Kollath,[5] E. Orignac,[6] R. Citro,[7] and T. Giamarchi[4]

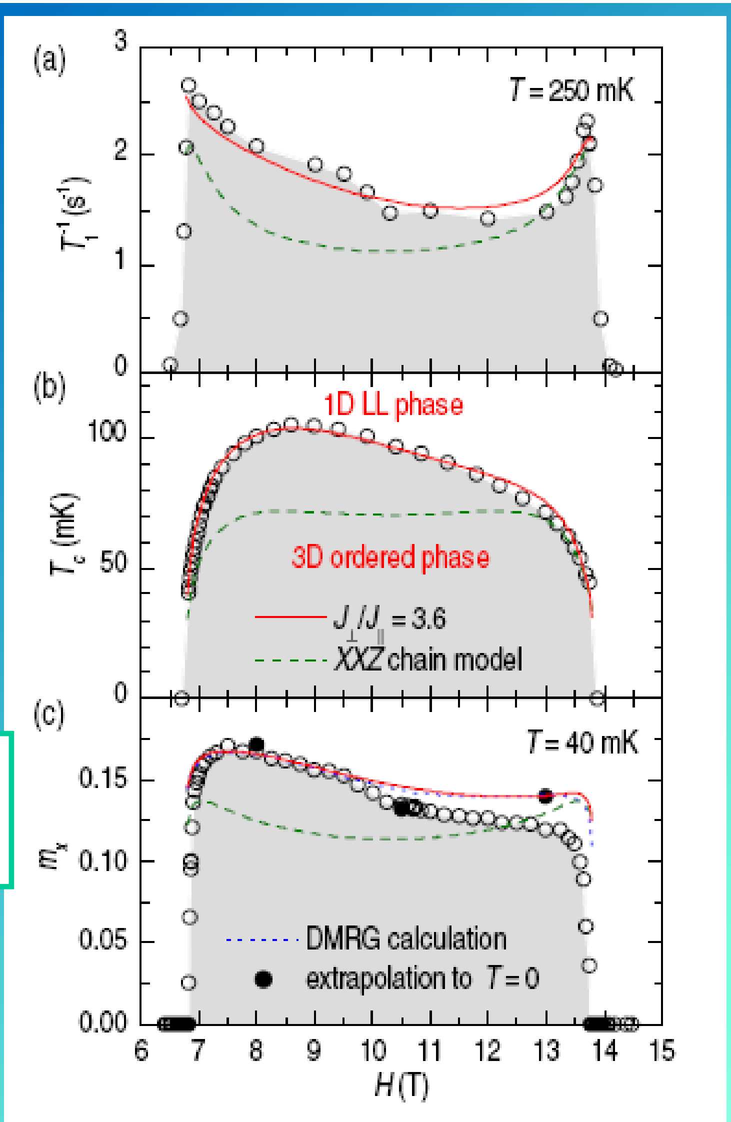


A ladder formed by the supposed exchange interactions between 1/2 spins of copper ions in the crystal of BPCB

$$T_1^{-1} = \frac{\hbar\gamma^2 A_\perp^2 A_0^x}{k_B u}\cos\left(\frac{\pi}{4K}\right)B\left(\frac{1}{4K}, 1-\frac{1}{2K}\right)\left(\frac{2\pi T}{u}\right)^{(1/2K)-1},$$

$$T_c = \frac{u}{2\pi}\left[\sin\left(\frac{\pi}{4K}\right)B^2\left(\frac{1}{8K}, 1-\frac{1}{4K}\right)\cdot\frac{zJ'A_0^x}{2u}\right]^{\frac{2K}{4K-1}}$$

$$m_x = F(K)\sqrt{A_0^x}\left(\frac{\pi z J' A_0^x}{2u}\right)^{1/(8K-2)},$$



First quantitative check of Luttinger liquid model!

Effects of disorder:
The quasiperiodic optical lattice

In collaboration with

Anna Minguzzi, LPMMC Grenoble, FR

Luis Santos & Xialong Deng, Hannover, DE

Edmond Orignac, ENS, Lyon, FR

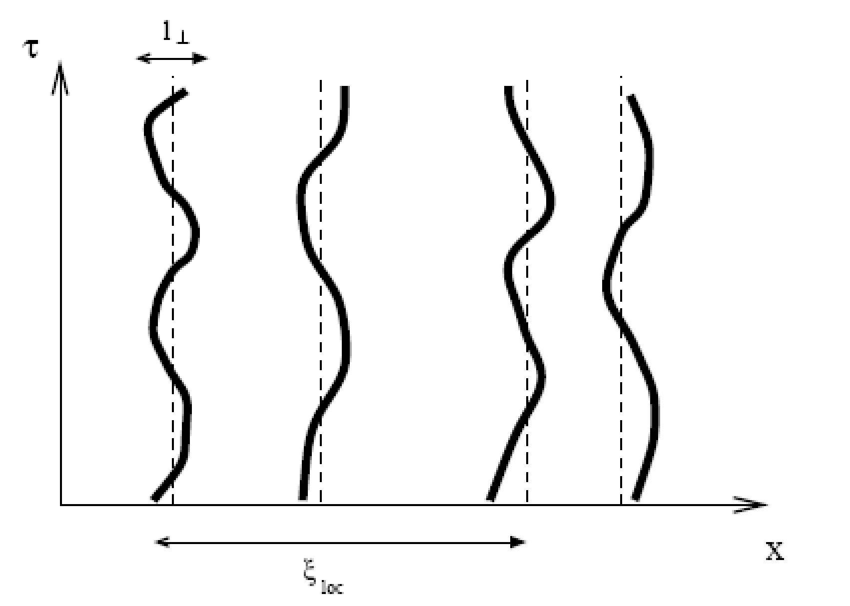
What's new between disorder and interactions

Free electrons are always localized

1D

$$\xi_{\text{loc}} \approx l$$

Exact solution (Berezinskii, 1974; Abrikosov, 197



Disorder rends electrons diffusive rather than ballistic!

(Altshuler and Aronov, 1985)

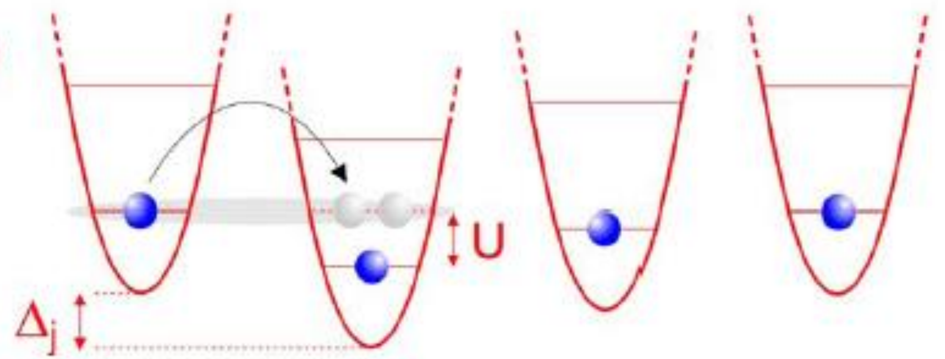
In reduced dimensions…strong reinforcement of disorder due to quantum fluctuations!

Competion of Anderson localization vs delocalization!

Interacting bosons in disordered potentials

For a system defined on a lattice one can derive a zero T model in which particles occupy the fundamental vibrational state:The Bose-Hubbard

$$\hat{H} = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) + \sum_j \epsilon_j \hat{n}_j$$

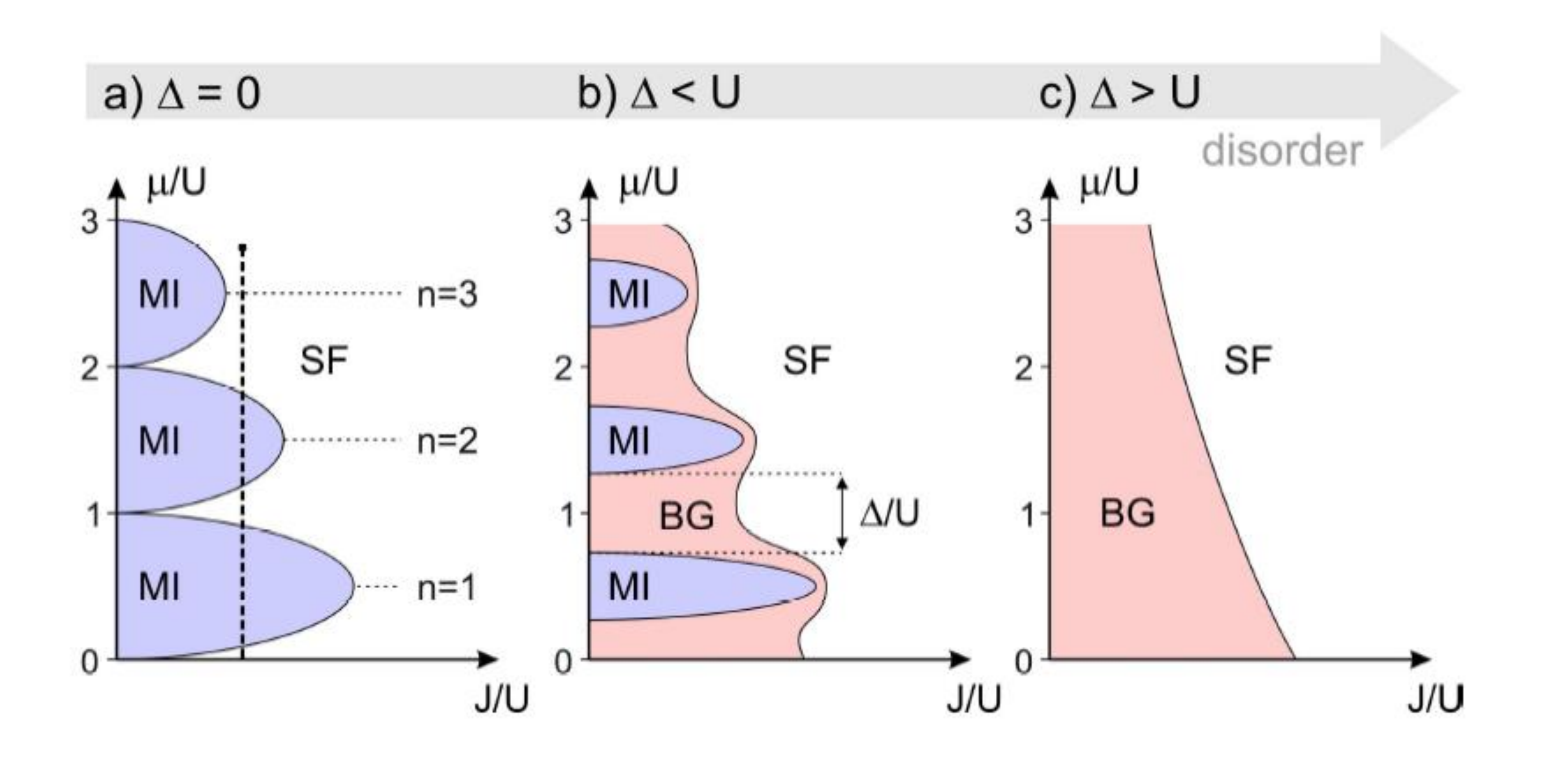


$$\epsilon_j \in [-\Delta/2, \Delta/2]$$

Fisher 1989, Jaksch 1998

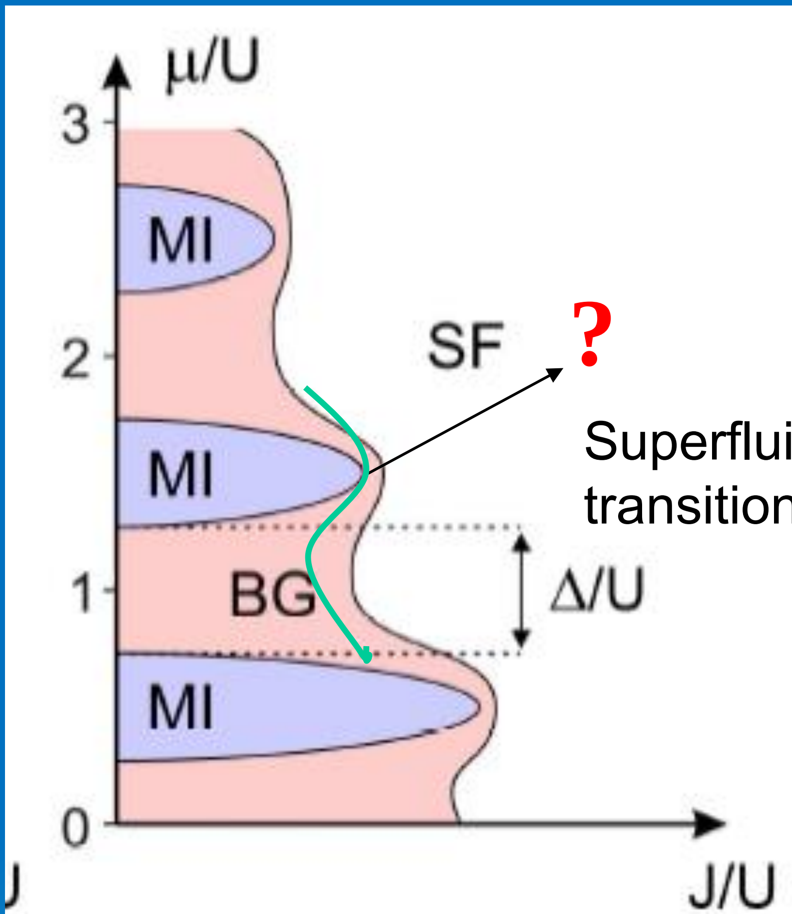
hopping energy	interaction energy	disorder
J	**U**	$\Delta = s_2/s_1$

Phase diagram of disordered bosons

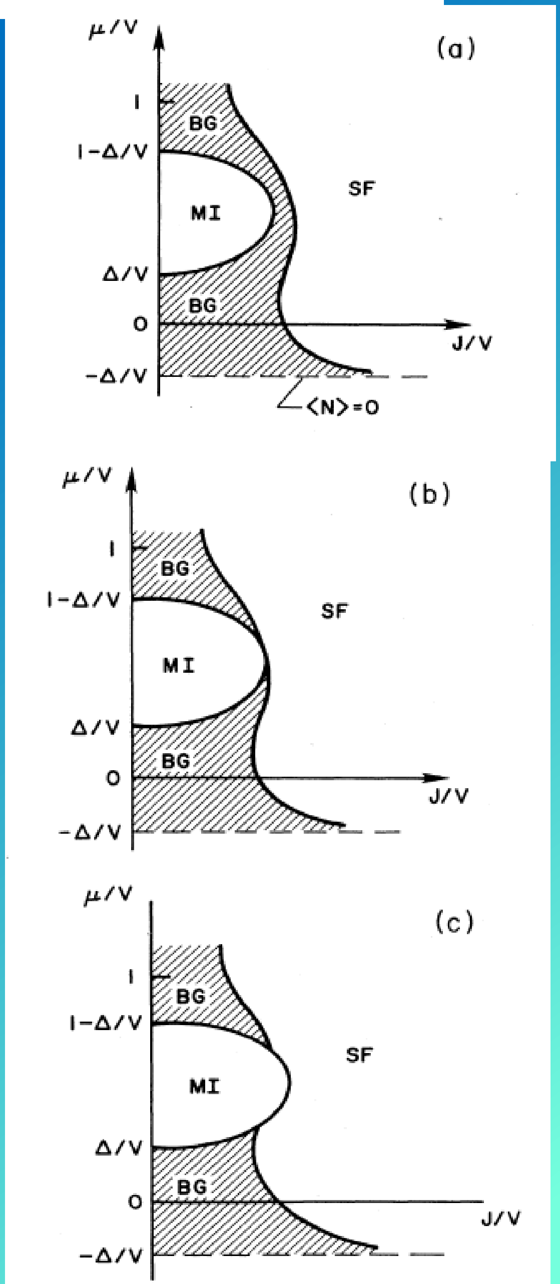


A new quantum phase appears: The **Bose-Glass phase (compressible but non-superfluid)** (Giamarchi &Shulz, 1988, Fisher, 1989)

Direct SF-MI phase transition? One of the possible Fisher scenarios



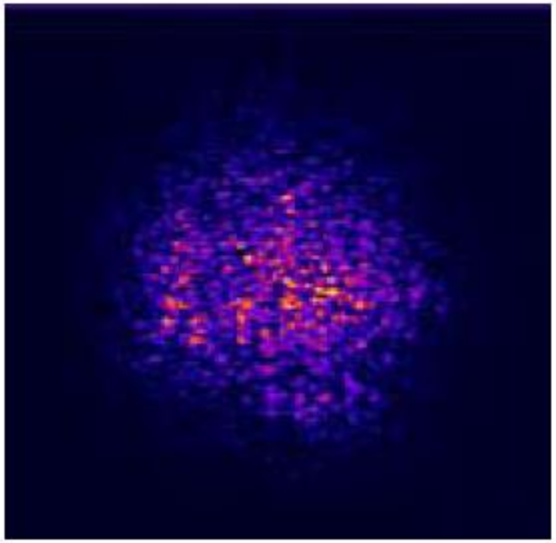
Superfluid line transition



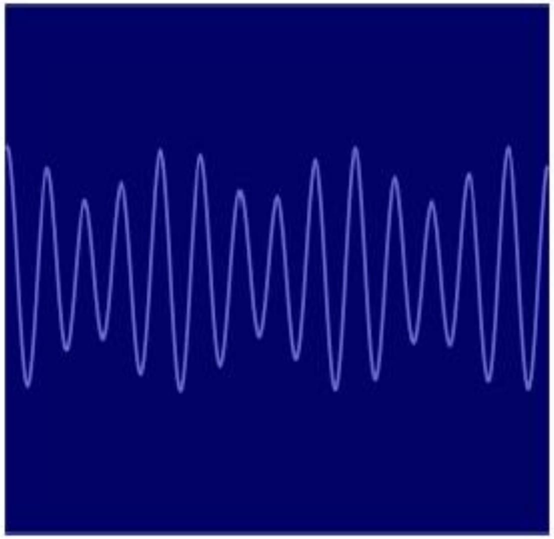
M.A. Fisher, PRB 1989

Disordered Optical Potential

speckle pattern



bichromatic lattice



- ✓ random potential
- ✗ large length scale in our set-up (several μm)

- ✓ quasiperiodic potential
- ✓ smaller length scale (1 μm or less)

J.E.Lye et al. PRL **95**, 070401 (2005)

C. Fort et al. PRL **95**, 170410 (2005)

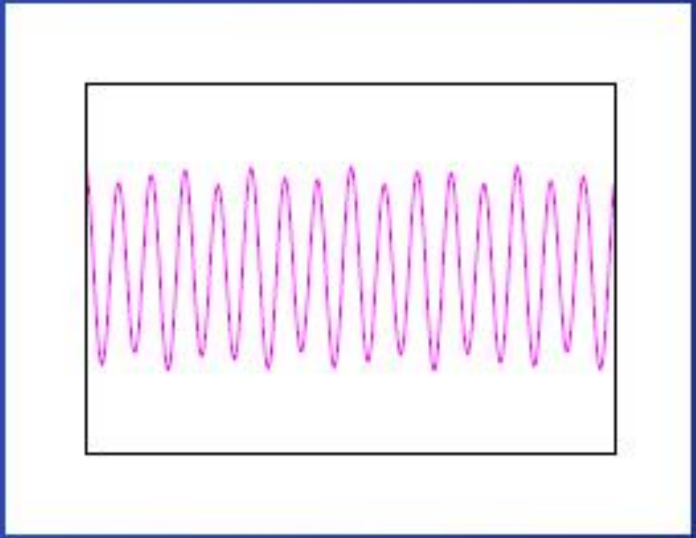
Non-periodic modulation of the energy minima with length scale

$$d = \left(\frac{2}{\lambda_1} - \frac{2}{\lambda_2}\right)^{-1}$$

Quasiperiodic potentials

- A pseudo-disorder for studying Anderson localization
- Experiments on bichromatic optical lattices (LENS, Florence):

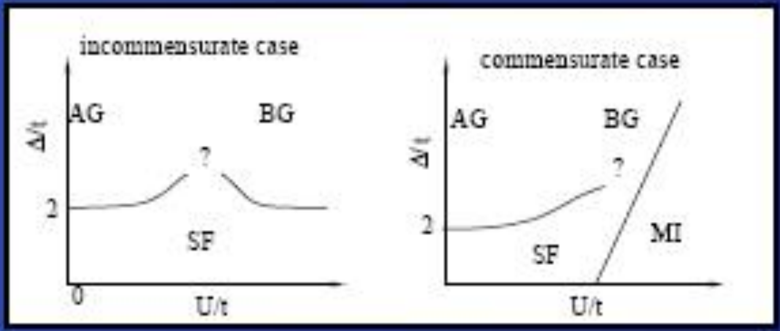
$$V(x) = V_1 \cos^2(k_1 x) + V_2 \cos^2(k_2 x)$$



important parameters: $\beta = k_2/k_1$, $\Delta = V_2/V_1$

Some questions

- Noninteracting limit: for irrational β, extended to Anderson-localized transition at finite $\Delta = V_2/V_1$ *[Aubry-André model, Harper model, almost Mathieu problem]*. And if β is rational?
- what happens in presence of repulsive interactions?



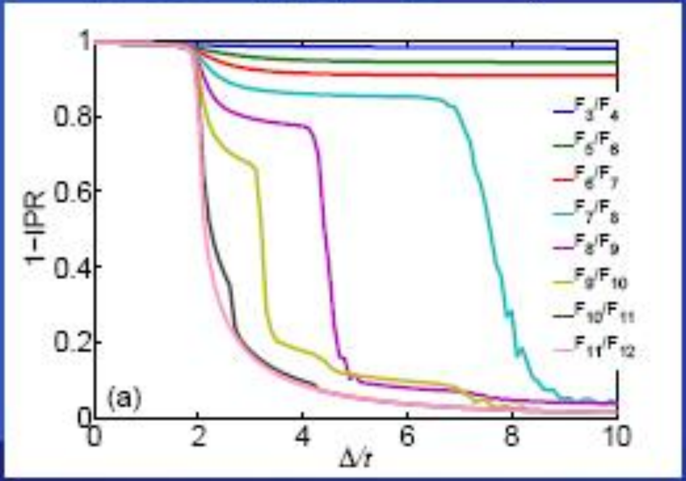
* repulsions are in competition with localization

* how does disorder affect the Mott insulator phase?

The Fibonacci example

- Localization "transition" for rational values of β?
  For $U=0$ use inverse participation ratio
  $IPR=\sum_i |\psi_i^{(0)}|^4$ (localized: IPR=const, extended: $IPR=O(1/N)$). Related to number fluctuations: $\langle \Delta n^2\rangle/N = 1 - IPR$

- Fibonacci sequence $F_1=1$, $F_2=1$, $F_{n+1}=F_n+F_{n-1}$, $\lim_{n\to\infty} F_{n+1}/F_n = (1+\sqrt{5})/2$.

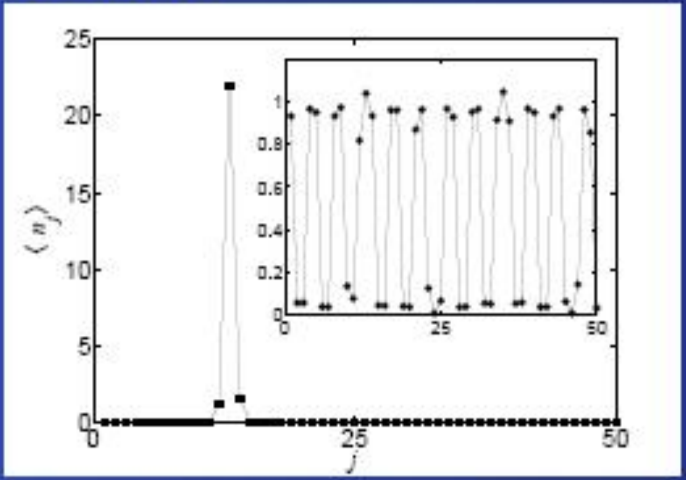


The transition is sharper as n increases

EPJ B, **68,** p.435(2009)

Disorder + interactions

- the Bose glass is a compressible, but non-superfluid phase *[Giamarchi and Schulz (1988), Fisher et al (1989)].* $U \to \infty$ (TG) : Anderson localization of the mapped Fermi gas *[Graham et al (2005)].*
- Anderson vs Bose glass: interactions make the density profile rather uniform



Lattice model

- Hamiltonian
  $H = -t\sum_i (b_i^\dagger b_{i+1} + h.c.) + (U/2)\sum_i n_i(n_i - 1) + \Delta\sum_i \cos(2\pi\alpha i)n_i$
- Ground state properties: efficient DMRG algorithm (Xiaolong Deng, LPMMC)
- Observables:
  - Superfluid fraction $f_s = \frac{N_{sites}^2}{Nt\theta^2}\left(E_\theta^N - E_0^N\right)$
  - Compressibility
    $\chi^{-1} = L[E(N+1) + E(N-1) - 2E(N)]$

Low energy description: Luttinger liquid

$\Delta \ll 2t$ Luttinger Liquid

Haldane, PRL 47 (1981)

$$H_0 = \frac{1}{2\pi}\int dx \left(\frac{v_s}{K}[\nabla\phi(x)]^2 + v_s K[\pi\Pi(x)]^2 \right)$$

$$[\phi(x), \Pi(x')] = i\delta(x - x')$$

$$\rho(x) = \left(\rho_0 - \frac{1}{\pi}\nabla\phi(x)\right)\sum_{p=-\infty}^{\infty} e^{i2p[\pi\rho_0 x - \phi(x)]}$$

$\theta = \pi \int^x dx' \Pi(x') =$ Superfluid phase

$\partial_x \phi =$ density (dual variable)

Linearized quantum hydrodynamics

Correlation functions

$$\langle e^{in\theta(x)} e^{-in\theta(x')}\rangle = \left(\frac{\alpha}{|x - x'|}\right)^{\frac{n^2}{2K}}$$

$$\langle e^{i2m\phi(x)} e^{-i2m\phi(x')}\rangle = \left(\frac{\alpha}{|x - x'|}\right)^{2m^2 K}$$

- $K \searrow$ as repulsion $\nearrow$.
- $K = \infty \rightarrow$ Free Bose gas
- $K = 1 \rightarrow$ Free Fermi / Tonks-Girardeau gas.

Perturbative treatment of disorder

$$H_{bl} = V_i \int dx \cos(2k_i x)\rho(x)$$

$$= \sum_{p=-\infty}^{\infty} \frac{\rho_0 V_i}{2} \int dx \cos[(2\pi p\rho_0 \pm 2k_i)x - 2p\phi(x)].$$

- Luttinger liquid (superfluid) behavior persists provided that the filling is not commensurate $p\rho_0 \pm k_i/\pi \in \mathbb{Z}$ (not satisfied)

- Under RG the operator is irrelevant if $K > K_c = 2/p^2$

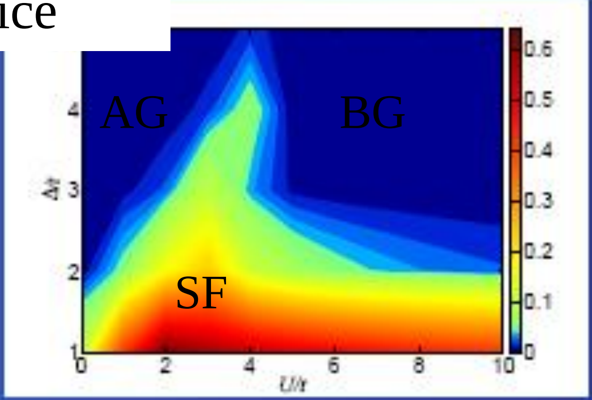
- Different from random distributed disorder! $K_c = 3/2$ (Giamarchi, Schulz PRB (1998))

Superfluid towards a Mott-insulating transition

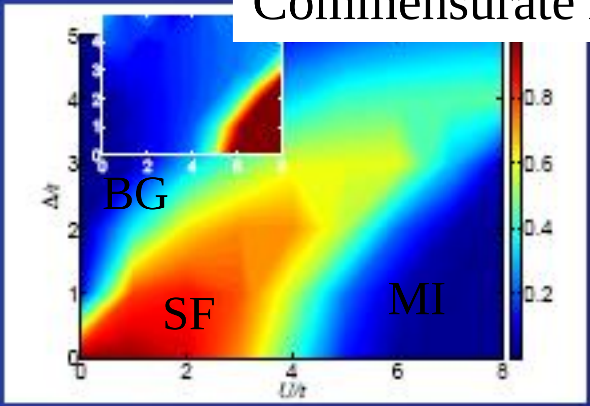
Phase diagram from DMRG

- Superfluid fraction and inverse compressibility for an interacting 1D Bose gas in a quasiperiodic lattice

Incommensurate lattice



Commensurate lattice

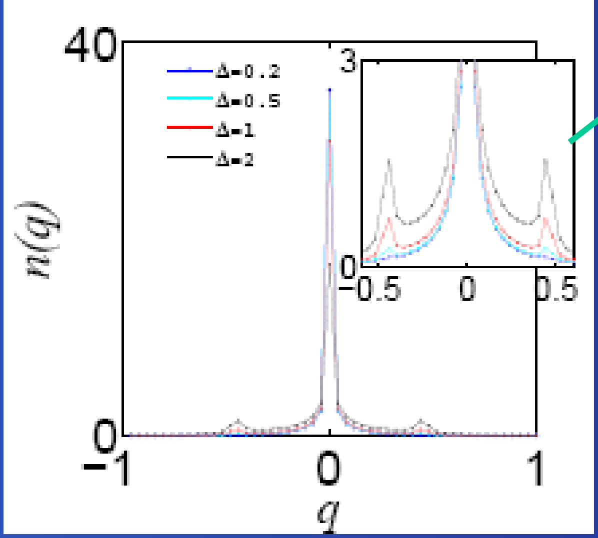


[Xiaolong Deng, R. Citro, AM and E. Orignac, PRA (2008)], related work *[G. Roux et al PRA (2008)]*

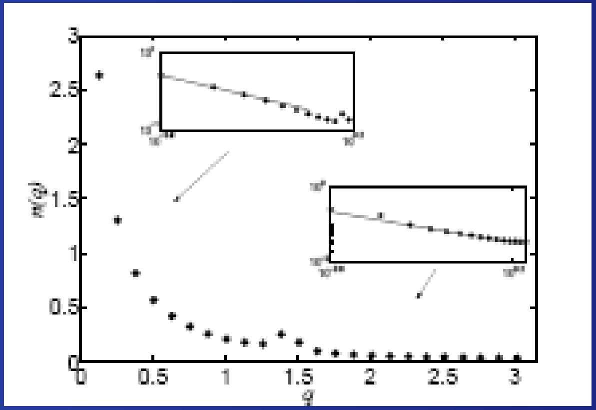
Momentum distribution

- In the superfluid phase, Luttinger liquid description: interactions broaden the momentum distribution peaks, power-law structure $n(q) \propto q^{1/2K-1}$

Diffraction of ground-state wf



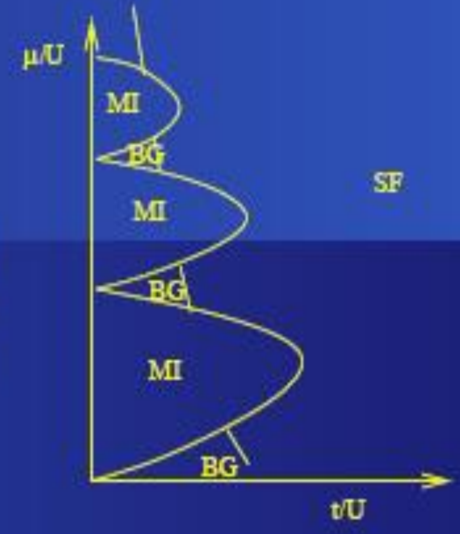
$\pm Q = \pm \frac{2\pi}{a}(1-\alpha)$ $\alpha = k_2/k_1$



$$n(q) = \mathcal{N} \sum_{lm} \mathrm{e}^{iq(l-m)a} \langle b_l^\dagger b_m \rangle.$$

U=2t, N=50

Mott lobes

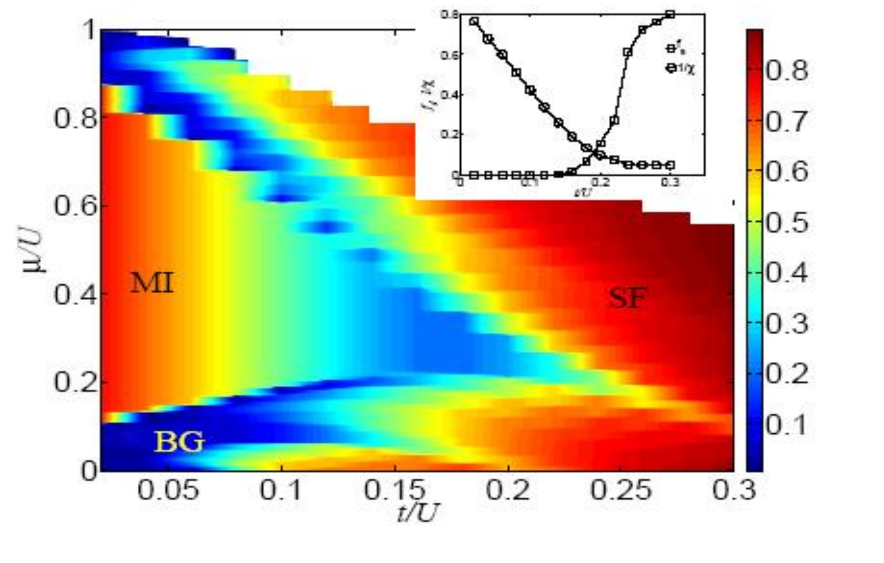


- Possibility for a direct superfluid-Mott insulator transition? One of the Fisher scenarios
- NO for true disorder…

  but YES for quasiperiodic potentials: DMRG

Phase Diagram: Commensurate case <n>=1

$N = N_{sites} = 20$



Main messages

We showed evidence for a rich phase diagram for a one-dimensional Bose gas in a disordered lattice: emergence of a Bose glass

We provided prediction for a direct MI-SF transition and on the behavior of the momentum distribution

Outlook

Experimental probes: e.g. transport properties and evidence of Bose-glass behavior in the cloud expansion (e.g. L. Tanzi, PRL 111 (2013))

Effect of dissipation and particle losses for systems beyond cold atoms

Outlook

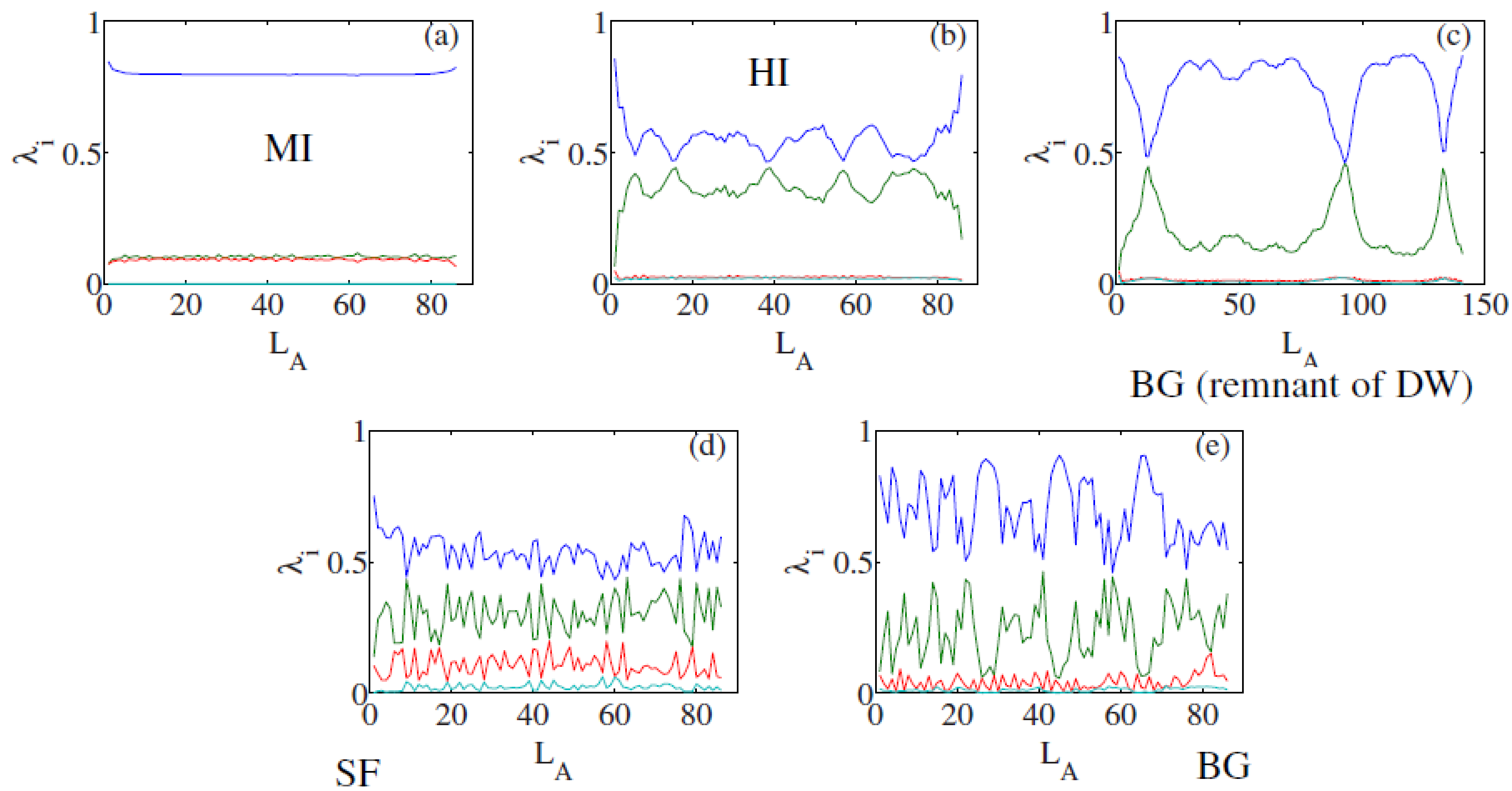
Experimental probes: e.g. transport properties and evidence of Bose-glass behavior

Temperature effects and Bose-glass collapse

Effect of dissipation and particle losses for systems beyond cold atoms

Thank you!

The entanglement spectrum: behavior of largest eigenvalues



$U/t = 5$ and $L = 89$.

X. Deng et al. New Jour. Phys., 15 (2013) 045023

DMRG for the quasiperiodic system

We consider a system with periodic boundary conditions and use the infinite-size algorithm to build the Hamiltonian up to the length L

the Hilbert space of bosons is infinite; to keep a finite Hilbert space in the calculation, we choose the maximal number of boson states approximately of the order 5*n*, varying *nmax* between *nmax=6 and 15*, except close to the Anderson localization phase where we choose the maximal boson states *nmax=N*.

The number of eigenstates of the reduced density matrix are chosen in the range 80–200.

To test the accuracy of our DMRG method, in the case U=0 or for finite U and small chain, we have compared the DMRG numerical results with the exact solution obtained by direct diagonalization

The calculations are performed in the canonical ensemble at a fixed number of particles N.

DMRG for the quasiperiodic system

We have studied the extended Bose–Hubbard model using the DMRG method with open boundary conditions [32–36]. The considered system sizes range up to 233 sites and we have taken up to 60 disorder realizations per point. In order to avoid the presence of metastable states we allow the number of optimal states to shrink or expand at every DMRG step according to a two-step algorithm. The algorithm keeps at least one of the eigenvectors in the blocks of the reduced density matrix[6] if they have only zero eigenvalues, and then keeps an additional eigenvector with zero eigenvalue in the block with non-zero eigenvalues if they decay very sharply to zero. The number of sweeps in the DMRG is 12 for weak disorder and up to 20 for strong disorder. Furthermore, we have eliminated the edge states in the HI phase by adding one more particle or by coupling two extra hard-core bosons at the edges of the chain in order to form a singlet state [37].

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Dynamics of correlations in TLL

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Interacting bosons in quasi-period lattices

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Dipolar gases In one-dimensions